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GLOBAL SOLUTION FOR NONCONVEX QUADRATIC PROGRAMMING

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Abstract

This paper concerns with an exact penalty function method to convert constrained optimization problem into a single unconstrained optimization problem. The use of the interval analysis method was employed to find the global solution in nonconvex quadratic programming.

Key words

Unconstrained optimization; Exact penalty functions; Nonconvex quadratic programming.

I. Introduction

Global optimization concerns with characterization and computation of global maxima or minima of nonlinear function.

The problem of designing algorithms that obtain global solution is very difficult, since in general there is no criterion for deciding whether a local solution is global.

Effort has been made to introduce some basic tools and techniques of interval analysis which are approved to find global solution in nonconvex quadratic programming.

Consider the following nonconvex quadratic programming problem.

$$(QP) \quad \text{Min } f(x) = x^T Qx - r^T x$$

$$\text{s.t. } g_i(x) = a_i^T x - b_i \le 0, \quad i = 1,2,\ldots,m$$

$$x \in X,\ X \subset R^n \text{ is a compact set.}$$

where Q is indefinite matrix, x, r $\in R^n$, $b \in R^m$, $a_i \in R^n$

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The corresponding exact penalty problem is

$$(QP\mu) \quad \underset{x \in X}{\text{Min}} \quad f(x) + \mu\, p(x),\ \mu > 0.$$

where $P(x) = \sum_{i=1}^{m} \max(0, g_i(x))$ is a penalty function.

In the section 2 interval arithmetic operations since these are quite different from the real arithmatic operations.

In section 3 inclusion functions and theorems are employed with example.

Examples in section 4 and use of algorithm of interval analysis method was made to find global solution of nonconvex quadratic programming.

II. INTERVAL ARITHMETIC OPERATIONS

To understand the interval analysis refer to (2) and (3). In this section some basic tools and techniques of interval analysis are introduced as applied to the global optimization problem.

Let I be the set of real compact intervals [a, b], a, b ∈ R.

Operations in I are define by the expression

$$A * B = \{a * b : a \in A,\ b \in B\} \text{ for } A, B \in I \qquad (2.1)$$

Where the symbol * stands for sum, difference, product and division, and where for the moment, A / B is only defined if 0 ∈ B. If $\alpha \in A$ and $\beta \in B$ then from (2.1) it follows that

$$\alpha * \beta \in A * B$$

which is called inclusion principle of interval arithmetic.

Moore (1966) proved that (2.1) is equivalent to the following constructive rules:

$$\begin{aligned} &[a, b] + [c, d] = [a + c, b + d], \\ &[a, b] - [c, d] = [a - d, b - c], \qquad (2.2) \\ &[a, b] \,.\, [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad\ bc, db)] \\ &[a, b] / [c, d] = [a, b] \,.\, [1/d, 1/c] \text{ if } 0 \in [c, d] \end{aligned}$$

The property (2.2) is one of the main differences between interval

arithmetic and real arithmetic and shows that the subtraction and division in I are not the inverse operations of addition and multiplication respectively as is the case in R. For example,

$$[0, 1] - [0, 1] = [-1, 1]$$

$$[1, 2] / [1, 2] = [1/2, 2]$$

Another main difference is given by the fact that the distributive law of real arithmetic is not valid in general. Only the so-called sub distributive law,

$$A(B + C) \subseteq AB + AC \qquad \text{for } A, B, C \in I$$

holds in I. For example:

$$[0, 1] \cdot [1 -1] = 0$$

$$[0,1] \quad 1 - [0, 1]\ 1 = [-1, 1]$$

The set of real number is denoted by R and the set of real of compact intervals [a, b], a, b ∈ R by I. If A ∈ I then we write A = [ℓbA, ubA] denoting the lower and upper boundaries of A by ℓbA and ubA. Intervals of the form [a, a] are called point intervals. These are identified with the corresponding reals. The width of an interval A = [a, b] is denoted by

$$w(A) = b - a$$

and the mid point by

$$\text{mid } A = \frac{a+b}{2}$$

The set of real m-dimensional column vectors is denoted by R^m and the set of m-dimension interval column vectors by I^m. The ^{T} means the transpose of a row or a column.

If $A = (A_1, \ldots, A_m)^T \in I^m$ then A is a right parallelepiped $A_1 \times A_2 \times \ldots \times A_m$. The vector of left endpoints of A is denoted by $\ell b\, A = (\ell bA1, \ldots, \ell bA_m)^T$ and the vector of right endpoints is denoted by $ubA = (ubA_1, \ldots, ubA_m)^T$.

If $A = (A_1, \ldots, A_m)T \in I^m$ then the width of A is defined to be

$$w\ (A) = \max\ \{w(A_i);\ i = 1, \ldots, m\}.$$

and the midpoint of A is

$$\text{mid } (A) = (\text{mid } A_1, \ldots, \text{mid } A_m)^T.$$

The set of n x m real matrices is denoted by R^{nXm} and the set of n x m intervals matrices by I^{nXm}. If $A = (A_{ij}) \in I^{nXm}$ then

$$\text{mid } A = (\text{mid } A_{ij}) \in R^{nXm}$$

is the midpoint of A.

If $A, B \in I$, $A, B \in I^m$ or $A, B \in I^{n \times m}$ then

$$A \cap B = \{x : x \in A \text{ and } x \in B$$

is the intersection of A and B

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

is the union of A and B.

For $D \subseteq R^m$ we denote I(D) the set of all boxes $Y \in I^m$ with $Y \subseteq D$.

For example, if $x \in I^m$, and thus x R^m, the set of all sub boxes Y of X just I(X).

III. INCLUSION FUNCTIONS.

. In the treatment of optimization problems using interval arithmetics, the main tool is the concept and the application of inclusion functions

Let $D \subseteq R^m$ and f: D $\rightarrow$ R. Let further more $\square f(Y) = \{f(x): x \in Y\}$

be the range of f over $Y \in I(X)$.

A function F: I(D) $\rightarrow$ I is called an inclusion function for *f* if

$$\square f(Y) \subseteq F(Y) \text{ for any } Y \in I(X).$$

The interval function F:I(D) $\rightarrow$ I defined by F(y) = *f* (y) is an inclusion function for f

Example: If $f_1(x) = x - x^2$ and $f_2(x) = x(1-x)$ then $f_1(x)$ and $f_2(x)$ are different expressions but equal as a functions. Further, $f_1(y)$ and $f_2(y)$ are also different functions.

if y = [0, 1] then $f_1(y) = y - y^2 = [-1, 1]$ and $f_2(y) \doteq y(1 - y) = [0, 1]$.

For comparison $\square f(y) = [1, 1/4]$

THEOREM 3.1 Let $X \subseteq R^m$ be a box, let *f*: X $\rightarrow$ R be any function, and let F: I(X) $\rightarrow$ I be an inclusion functions for f. Then the global "minimum" f * = inf (*f* (X)) exists and

$$f^* \in F(Y)$$

holds for all leading boxes Y of the algorithm.

LEMMA 3.1 Let $(y_n)_{n=1}^{\infty}$ be a sequence. Then $w(Y_n) \to 0$ as $n \to \infty$.

THEROM 3.2 If the inclusion function F for f staisfies $w(F(Y)) \to 0$ as $w(Y) \to o$ then the sequence $(F(Yn))_{n=1}^{\infty}$ converges to the global minimum f^* i.e. the sequence $(y_n))_{n=1}^{\infty}$ convergence to f^*.

To understand above theorems and Lemma refer to [1].

IV. THE ALGORITHM

Now interval analysis method to find global solution for unconstrained optmization problem (UP). The unconstrained optimization problem.

$$(UP) \qquad \min_{x \in R^n} f(x);$$

where f(x) be a continous funcion.

ALGORITHM

Step 0. Set Y=X

Step 1. Calcualte F(Y) and $\overline{f} = ubF(c)$ *wher* $c = mid\ Y$.

Step 2. Set y = lbF (Y).

Step 3. Initialize list L = {(Y, y)}.

Step 4. Choose a coordinate direction k parallel to which y has an dege of maximum length, that is , $k \in \{i: w(y_i) = w(Y)\}$

Step 5. Bisect normal to direction k getting boxes V_1, V_2 such that $Y = V_1 \cup V_2$.

Step 6. Calclulate $F(V_1)$, $F(V_2)$

Step 7. Set V_i = lbF (V_i), for i = 1, 2

Step 8. Enter the pairs (V_1, V_1) and (V_2, V_2) at the end of the list.

Step 9. Chose a pair $(\overline{Y}, \overline{y})$ of the list which satisfies $\overline{y} \leq z$ for all pairs (Z,z) of the list.

Step 10. Discard all pairs (Z, z) from the list that satisfy $\overline{f} < z$ (mid point test).

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Step 11. If termination criteria hold goto 14.

Step 12. Denote the first pair of the list by (Y, y). Set c = mid Y and
= min (, ubF(c)).

Step 13. Go to 4

Step 14. End.

EXAMPLE:

$$\text{Max } f(x) = 4x_1 - x + 2x_2$$

(P_1)

$$\text{s.t} \quad x_1 + x_2 \quad 1$$

$$x_1, \; x2 \quad 0.$$

The exact penality problem corresponding to (p_1) is:

Max $4x_1 - x + 2x_2 + \mu$ [max $(o, x_1 + x_2 - 1) +$ max $(0, -x_1)$ + max $(0, -x_2)$]]

By opplying the algorithm, the obtained global solution in the talble 1.

Table 1

Algorithm	
Y_0	[-1, 1] x [-0.5, 1]
N	65
	0.000001
	100.00
F*	[2.8460623, 2.8552153]
Y*	[-0.85423, 0.816496] x
	[0.07643, 0.183503419]

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