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AN ASYNCHRONOUS PARALLEL NONLINEAR MULTISPLITING RELAXATION METHODS

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Abstract

In this paper, we set up an asynchronous parallel nonlinear multisplitting relaxation methods for solving system of nonlinear equation. With special choices of the relaxed parameters in the new methods, not only can the convergence properties of them be improved, but also many applicable and efficient asynchronous parallel nonlinear multisplitting iteration methods; such as the Jacobi, Gauss-Seidel, SOR as well as the asynchronous parallel nonlinear multisplitting AOR-Newton, -Cord and -Steffensen programs, etc., can be obtained Under proper conditions, we build convergence theories about these asynchronous methods, and estimate their asymptotic convergence rates in detailed manner.

Introducation

Consider the large-scale system of nonlinear equations:

$$F(x) = O, \quad F : D \subset R^n \text{------>} R^n \tag{1.1}$$

Given α ($\alpha \leq n$, an integer) nonempty subsets J_i ($i = i, 2, \ldots, \alpha$) of the set $\{1, 2, \ldots, n\}$ with

$$\bigcup_{i=i}^{\alpha} J_i = \{1, 2, \ldots, n\},$$

where $J_1, J_2, \ldots, J_\alpha$ may overlap among them, for each $i \in \{1, 2, \ldots, \alpha\}$, we assume that

(a) $f^{(i)}: D \times D \subset R^n \times R^n$ ------> R^n satisfies

$$f^{(i)}(x;x) = \left(f_1^{(i)}(x;x),\ f_2^{(i)}(x;x),\ \ldots,\ f_n^{(i)}(x;x)\right)^T = F(x), \ \forall\ x \in D;$$

(b) $E_i = \text{diag}\left(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}\right) \in \overset{T}{L}(R^n)$ satisfies

$$\begin{cases} e_m^{(i)} = \begin{cases} e_m^{(i)} \geq 0, \text{ for } m \in J_i, M = 1(1)n \\ 0 \quad \text{otherwise.} \end{cases} \\ \sum_{i=1}^{\alpha} E_i = I \ (I \in L(R^n) \text{ is an identity matrix}) \end{cases}$$

Then, the collection of pairs $(f^{(i)}, E_i)$, $i = 1, 2, \ldots, \alpha$, is called a nonlinear multisplitting of the mapping $F: D \subset R^n$ -----> R^n.

Based on this concept, Frommer first proposed [1] the parallel nonlinear multisplitting iteration methods for getting the solution of the system of nonlinear equation (1.1) in 1989, these methods are convenient for computing and are of much strong function of parallel computation. Therefore, they afford basis for designing synchronous parallel iterative methods in the sense of nonlinear multisplitting for solving nonlinear problem (1.1).

In order to overcome the intrinsic shortcomings of the synchronous algorithms and make the practical computing efficiency of the multiprocessor systems yield well in this paper, aiming at the concrete characteristics of the MIMD-systems, we set up an asynchronous parallel nonlinear multisplitting relaxation methods for solving system of nonlinear equation (1.1). These methods are of the properties of conveniently computing and freely and flexibley communicating, thus, they are much suitable to execute in the asynchronous parallel environments. With special choices of the relaxed parameters, not only can the convergence properties of the new asynchronous parallel methods be improved, but also many applicable and efficient asynchronous parallel nonlinear multisplitting iteration methods such as the jacobi, Gauss-Seidel, SOR and so on can be obtained. In addition, the asynchronous parallel nonlinear multisplitting AOR- Newton, -Chord and -Steffensen programs, being of highly practical value, are given, too, and the effectiveness, applicability and flexibility of the new methods are therefore further strengthened. Under proper conditions, we build convergence theories about the asynchronous parallel nonlinear multisplitting accelerated overrelaxation methods, and estimate their asymptotic convergence rates in a detailed manner. This work is also further generalization of those of [10] to system of nonlinear equation.

2. Nonlinear Multisplitting Relaxation Methods

Suppose the multiprocessor system discussed in made up of α CPU's, we initially introduce the following notations.

(1) $\forall\ p \in N_o := \{0, 1, 2, ...,\}$, $J = \{J(p)\}_{p \in N_o}$ is used to denote a nonempty subset of the set $\{1, 2, ..., \alpha\}$;

(2) $S = \{s_1(p)\ s_2(p), ..., s_\alpha(p)\}_{p \in N_o}$ are α infinite sequences. The sets J and S have the following properties:

(a) $\forall\ i \in \{1, 2, ..., \alpha\}$, the set $\{p \in N_o \ / \ i \in J(p)\}$ is infinite:

(b) $\forall\ i \in \{1, 2, ..., \alpha\}$, $\forall\ p \in N_o$ there holds $s_i(p) \le p$;

(c) $\forall\ i \in \{1, 2, ..., \alpha\}$, there holds $\lim_{i \to \infty} s_i(p) = \infty$

If $\forall\ p \in N_o$, we define

$$s(p) = \min_i s_i(p),$$

there obviously hold

$$s(p) \le p, \ \lim_{p \to \infty} s(p) = \infty$$

Now, for the system of nonlinear equation (1.1), we consider the following asynchronous parallel nonlinear multisplitting AOR method:

Method-I

Suppose that we have got approximations $x^o, x^1, ..., x^P$ of the solution x^* of (1.1), then the (p + 1)-th approximation $x^{p+1} = (x_1^{P+1}, x_z^{p+1}, ..., x_n^{p+1})^T \in R^n$ of x^* can be calculated by:

$$x^{p+1} = \sum_{i=1}^{\alpha} E_i\, x^{i,p} \tag{2.1}$$

with $x^{i,p} = (x_1^{i,p}, x_2^{i,p} ..., x_n^{i,p})^T$ $(i = 1, 2, ..., \alpha)$ being

$$x^{i,p} = \begin{cases} \frac{\omega}{r}\, \tilde{x}^{i,p} + \left(1 - \frac{\omega}{r}\right) \cdot x^{s_i(p)}, & \text{for } i \in J(p) \\ x^p, & \text{for } i \notin J(p) \end{cases} \tag{2.2}$$

where

$$x^{s_i(p)} = \left(x_1^{s_i(p)}, x_2^{s_i(p)}, ..., x_n^{s_i(p)}\right)^T, \ i = 1, 2, ..., \alpha$$

while

$$\tilde{x}^{i,p} = \left(\tilde{x}_1^{i,p}, \tilde{x}_2^{i,p}, \ldots, \tilde{x}_n^{i,p}\right)^T, \quad i = 1, 2, \ldots, \alpha$$

is defined to be

$$\hat{x}_m^{i,p} = \begin{cases} r\, \hat{x}_m^{i,p} + (1-r)\, x_m^{s_i(p)}, & \text{for } m \in J_i \quad m = 1(1)n \\ x_m^{s_i(p)}, & \text{for } m \notin J_i \end{cases} \tag{2.3}$$

$\hat{x}_m^{i,p}$ $(m \in J_i,\ m = 1(1)n,\ i = 1, 2, \ldots, \alpha)$ are successively determined by the nonlinear equations,

$$f_m^{(i)}\left(x^{s_i(p)};\ \hat{x}_1^{i,p}, \ldots, \hat{x}_{m-1}^{i,p}, \hat{x}_m^{i,p}, x_{m+1}^{s_i(p)}, \ldots, x_n^{s_i(p)}\right)^T = o,$$

$$m \in J_i, \quad m = 1(1)\, n. \tag{2.4}$$

Here, $r \in (o, +\infty)$ is called relaxation factor while $\omega \in (o, +\infty)$ acceleration factor.

Making use of (2.3), $x^{i,p}$ defined by (2.2) can be equivalently expressed as

$$x_m^{i,p} = \begin{cases} \omega\, \hat{x}_m^{i,p} + (1-\omega)\, x_m^{s_i(p)}, & \text{for } m \in J_i,\ i \in J(p) \\ x_m^{s_i(p)}, & \text{for } m \in J_i,\ i \in J(p),\ m = 1(1)n. \\ x_m^{(p)}, & \text{for } i \notin J(p),\ i = 1, 2, \ldots, \alpha \end{cases} \tag{2.5}$$

so Method-I is practically proceeded with the formulas (2.1), (2.3)-(2.5), too.

From (2.1)-(2.4) we can easily see that when

$$\begin{cases} J(p) = \{1, 2, \ldots, \alpha\}, & \forall\ P \in N_0 \\ s_i(p) = p, & \forall\ i \in J(p),\ \forall\ P \in N_0 \end{cases}$$

Method-I becomes the known synchronous parallel nonlinear multisplitting AOR method [5, 10], while

$J(p) \subset \{1, 2, \ldots, \alpha\}$

it really describes an asynchronous parallel nonlinear multisplitting AOR method. Moreover, by (2.1) and (2.3)-(2.5) we know that by specially taking the relaxed parameter pair (r,ω) to be $(0,1)$ $(0, \omega)$, $(1, 1)$, $(1, \omega)$ and (ω, ω) in Method-I, the practical and efficient asynchronous parallel nonlinear multisplitting Jacobi, extrapolated jacobi, Gauss-Seidel, extrapolated Gauss-Seidel and SOR methods can be correspondingly generated.

In fact, the exact solution of the implicit nonlinear equation (2.4) is much difficult to obtain, so in concrete applications, we usually make use of known procedures to get an approximate soluation of (2.4).

Method-II

Suppose that we have got approximations $x^o, x^1, \ldots, x^p$ of the soluation x^* of (1.1) then the $(p+1)$-th approximation x^{p+1} is determined by (2.1)-(2.3) and

$$\hat{x}_m^{i,p} = x_m^{s_i(p)} - \frac{f_m^{(i)}\left(x^{s_i(p)};\ u_m^{i,p}\right)}{H_{mm}^{(i)}\left(x^{s_i(p)};\ u_m^{i,p}\right)},\quad m \in J_i,\ i \in J(p) \tag{2.6}$$

respectively. Where

$$u_m^{i,p} = \left(\tilde{x}_1^{i,p}, \ldots \tilde{x}_{m-1}^{i,p}, \hat{x}_m^{i,p}, x_{m+1}^{s_i(p)}, \ldots, x_n^{s_i(p)}\right)^T$$

$$i = 1, 2, \ldots, \alpha,\ m = 1(1)n.$$

and $H_{mm}^{(i)}(x;y)$ denotes the m-th diagonal element of an approximate matrix $H^{(i)}(x;y)$ of the matrix $\partial_2 f^{(i)}(x;y)$, while $\partial_2 f^{(i)}(x;y)$, $\partial_1 f^{(i)}(x;y)$ are the first order derivatives of $f^{(i)}(x;y)$about its variables y,x, respectively. In Method-II, if we particularly take:

(1) $H_{mm}^{(i)}\left(x^{s_i(p)};\ u_m^{i,p}\right)=\partial_2^{(m)} f_m^{(i)}\left(x^{s_i(p)};\ u_m^{i,p}\right)$, $m \in J_i$, $i \in J(p)$,

the asynchronous parallel nonlinear multisplitting AOR-Newton program can be obtained, as the nonlinear equation (2.4) is solved approximately by the Newton procedure. Here, $\partial_1^{(m)} f^{(i)}(x;y)$ and $\partial_2^{(m)} f^{(i)}(x;y)$represent the m-th diagonal elements of $\partial_1 f^{(i)}(x;y)$ and $\partial_2 f^{(i)}(x;y)$ individually for $m = 1(1)n$, $i = 1, 2, \ldots, \alpha$;

$$(2)\quad H^{(i)}_{mm}\left(x^{s_i(p)};\ u^{i,p}_m\right)=\frac{f^{(i)}_m\left(x^{s_i(p)};\ u^{i,p}_m+h^{i,p}_m e_m\right)-f^{(i)}_m\left(x^{s_i(p)};\ u^{i,p}_m\right)}{h^{i,p}_m},$$

$$m\in J_i,\quad i\in J(p),$$

the asynchronous parallel nonlinear multisplitting AOR-Chord program is got, since the nonlinear equation (2.4) is approximately solved by the Chord procedure. Here, $h^{i,p}_m$ ($m \in J_i$, $i \in J(p)$) are given difference step sizes, while $e_m \in R^n$ in the m-th unit vector.

(3):

$$H^{(i)}_{mm}\left(x^{s_i(p)};\ u^{i,p}_m\right)=\frac{f^{(i)}_m\left(x^{s_i(p)};\ u^{i,p}_m+f^{(i)}_m\left(x^{s_i(p)};\ u^{i,p}_m\right)e_m\right)-f^{(i)}_m\left(x^{s_i};\right)u^{i,p}_m}{f^{(i)}_m\left(x^{s_i(p)};\ u^{i,p}_m\right)},$$

$$m\in J_i,\quad i\in J(p),$$

the asynchronous parallel nonlinear multisplitting AOR-Steffensen program can be obtained, because the nonlinear equation (2.4) is now approximately solved by the Steffensen procedure.

Similarly, corresponding different choices of the parameter pair (r, w) in Method-II, we can also get an extensive sequence of asynchronous parallel nonlinear multisplitting accelerated over relaxation methods. For the length of the paper, we will n t enumerate them here one by one.

For the requirement of establishing convergence theories of the above two methods, we introduce an infinite sequence $\{m_\lambda\}_{\lambda\in N_0}$ according to the following rule:

m_0 is the least positive integer such that

$$\bigcup_{0\le s(p)\le p<m_0} J(p)=\{1,2,\ldots,\alpha\},$$

in general, $m_{\lambda+1}$ is the least positive integer such that,

$$\bigcup_{m_\lambda\le s(p)\le p<m_{\lambda+1}} J(p)=\{1,2,\ldots,\alpha\},\quad l=0,1,2,\ldots .$$

3. Prelimnary Knowledge

In the subsequent discussion, we will adopt the notations, concepts and basic facts used in [3-7]. Particularly, $\rho(.)$ and <.> are used to denote the spectral radius and comparison matrix of the corresponding matrix, respectively, while $|.|$ represents the absolute value of either a vector or a matrix.

Define nonnegative diagonal matrix sequences $\{I_p^{(1)}\}_{p\in N_0}$ and $\{I_p^{(2)}\}_{p\in N_0} \in L(R^n)$ to be

$$I_p^{(1)} = \sum_{i\in J(p)} E_i, \quad I_p^{(2)} = \sum_{i\in J(p)} E_i, \qquad p = 0, 1, 2, \ldots,$$

then in the light of [9] we know that the following conclusions hold.

Lemma 1: Let $x \in R^n$ be a positive vector. If sequence $\{\varepsilon^p\}_{p\in N_0} \in L(R^n)$ satisfies

$$|\varepsilon^{p+1}| \le I_p^{(1)} x + I_p^{(2)} |\varepsilon^p|, \; p = 0, 1, 2, \ldots,$$

then for any nonnegative integer $q \le p - 1$, there holds:

$$|\varepsilon^{p+1}| \le \Big(I - \prod_{j=p-q-1}^{p} I_j^{(2)}\Big) x + \prod_{j=p-q-1}^{p} I_j^{(2)} \, |\varepsilon^{p-q-1}|$$

Lemma 2: Let

$$I^{(o)} = \prod_{p=o}^{m_o-1} I_p^{(2)}, \quad I^{(\lambda+1)} = \prod_{p=m_\lambda}^{m_{\lambda+1}-1} I_p^{(2)}, \; \lambda = 0, 1, 2, \ldots,$$

then for any positive vector $x \in R^n$, there exists $\{\gamma^{(\lambda)}\}_{\lambda\in N_0} \subset [0,1)$, such that

$$I^{(\lambda)} x \le \gamma^{(\lambda)} x, \; \lambda = 0, 1, 2, \ldots, .$$

4. Convergence Analysis of Method-I

Assume that $x^* \in D$ is a soluation of system of nonlinear equation (1.1), and for each $i \in \{1, 2, \ldots, \alpha\}$, $f^{(i)}: D \times D \subset R^n \times R^n \dashrightarrow R^n$ is differentiable in a neighbourhood of $(x^*; x^*)$. We again introduce the following signs

$$\begin{cases} B_i = \partial_2 f^{(i)}(x^*;\ x^*) = (b_{mj})^{(i)},\ C_i = -\ \partial_1 f^{(i)}(x^*;\ x^*) = (c_m^{(i)}) \\ D_i = \text{diag}(B_i),\ i = 1,\ 2,\ \ldots,\ \alpha \end{cases} \tag{4.1}$$

While $L_i = (\lambda_{mj}^{(i)}),\ U_i = (u_{mj}^{(i)}) \in L(R^n)$ satisfy:

$$\lambda_{mj}^{(i)} = \begin{cases} -\ b_{mj}^{(i)}, & \text{for } m > j \text{ and } m,\ j \in J_i \\ 0, & \text{otherwise} \end{cases} \quad m,\ j = 1(1)n.$$

and

$$B_i = D_i - L_i - U_i, \tag{4.2}$$

respectively.

Evidently, $F : D \subset R^n$ -------> R^n is also differentiable in neighbourhood of $x^* \in D$ at this time, and there holds:

$$\begin{cases} F'(x^*) = \partial_1 f^{(i)}(x^*;\ x^*) + \partial_2 f^{(i)}(x^*;\ x^*) \\ \qquad = D_i - L_i - (U_i + C_i) = D - B,\ i = 1,\ 2,\ \ldots,\ \alpha \end{cases} \tag{4.3}$$

where,

$D = \text{diag}(F'(x^*),\ B = D - F'(x^*)$.

Therefore, while $\det(D_i) \neq 0$ $(i = 1, 2, \ldots, \alpha)$, $(D_i - L_i,\ U_i + C_i,\ E_i)$ $(i = 1, 2, \ldots, \alpha)$ is naturally resulted in a multisplitting of the matrix $F'(x^*) \in L\ (R^n)$.

Presently, we begin to set up local convergence theory of Method-I

Theorem 1: Let $x^* \in D$ be a solution of system of nonlinear equation (1.1), $(f^{(i)},\ E_i)$ $(i = 1, 2, \ldots, \alpha)$ be a nonlinear multisplitting of $F: D \subset R^n$ ------> R^n and $f^{(i)}$: $DXD \subset R^n\ X\ R^n$ ---> R^n be continuously differentiable in a neighbourhood of $(x^*;\ x^*)$ for each $i \in \{1, 2, \ldots, \alpha\}$. Suppose $F'(x^*) \in L\ R^n)$ be an H-martix and $(D_i - L_i,\ U_i + C_i,\ E_i)$ $(i = 1, 2, \ldots, \alpha)$ be a multisplitting of A.

$$\begin{cases} \langle F'(x^*)\rangle = |D_i| - |L_i| - |U_i + C_i| \\ \qquad = |D| - |B|,\ i = 1,\ 2,\ \ldots,\ \alpha \end{cases} \tag{4.4}$$

$$|x^{t+1}-x^*| = \sum_{i\notin J(t)} E_i \, |\mathscr{L}^{(i)}(r,\ \omega)| \ |x^{s_i(t)} - x^*| + \sum_{i\notin J(p)} E_i \ |x^t - x^*| + \sum_{i\notin J(p)} E_i \ |R^{(i)}(x^{s_i(t)})|$$

$$\leq \sum_{i\notin J(p)} E_i \ |\mathscr{L}^{(i)}(r,\ \omega)| \ |\delta \, x_\varepsilon + \sum_{i\notin J(p)} E_i \ |x^t - x^*| + \frac{\omega}{r}\varepsilon \ \delta \sum_{i\notin J(p)} E_i \ x_\varepsilon$$

$$\leq \sum_{i\in J(t)} E_i \ \sigma_\varepsilon \ \delta \ x_\varepsilon + \sum_{i\notin J(p)} E_i \ |x^t - x^*|,$$

that is

$$|x^{t+1} - x^*| \leq I_t^{(1)} \ \sigma_\varepsilon \ \delta \ x_\varepsilon \ \ I_t^{(2)} \ |x^t - x^*| \tag{4.23}$$

which implies

$$|x^{t+1} - x^*| \leq \delta \ x_\varepsilon \, ,$$

or in other words, $x^{t+1} \in N(x^*, \ \delta)$. In the light of the induction, we accomplish our test.

Part II: Suppose $x^{(o)} \in N(x^*, \ \delta)$, then

$$|x^p - x^*| \leq \Delta_\iota \ x_\varepsilon \, , \ \forall \ P \geq m_\iota, \quad \iota = 0, \ 1, \ 2, \ \ldots, \tag{4.24}$$

where,

$$\Delta_o = \left(\sigma_\varepsilon + (1-\sigma_\varepsilon) \ \gamma^{(o)}\right) \ \delta, \ \Delta_{\iota+1} = \left(\sigma_\varepsilon + (1-\ \sigma_\varepsilon) \ \gamma^{(\iota+1)}\right) \ \Delta_\iota, \ \iota = 0, \ 1,2, \ \ldots,$$

As a matter of fact, when $\iota = 0$, as $p \geq m_\iota$, by (4.23) and Lemma 1, 2 we can get

$$|x^{p+1} - x^*| \leq I_p^{(1)} \ \sigma_\varepsilon \ \ \delta \ x_\varepsilon + I_p^{(2)} \ |x^p - x^*|$$

$$\leq \left(I - \prod_{J=o}^{p} I_j^{(2)}\right) \ \sigma_\varepsilon \ \ \delta \ x_\varepsilon + \prod_{J=o}^{p} I_j^{(2)} \ |x^o - x^*|$$

$$\leq \left(\sigma_\varepsilon \, I + (1 - \sigma_\varepsilon) \prod_{J=o}^{p} I_j^{(2)}\right) \ \delta \ x_\varepsilon$$

$$\leq \left(\sigma_\varepsilon \, I + (1 - \sigma_\varepsilon) \ \ I^{(o)}\right) \ \delta \ x_\varepsilon$$

$$\leq \left(\sigma_\varepsilon \ + (1 - \sigma_\varepsilon) \ \ \gamma^{(o)}\right) \ \delta \ x_\varepsilon = \Delta_o \ x_\varepsilon.$$

Assume that when $p \geq m_\iota$, (4.24) have been proved. Then, when $p \geq m_{\iota+1}$, again, by making use of (4.16), (4.18), (4.20), (4.22), Lemma 1 as well as Lemma 2, we have the estimation

$$|x^{p+1}-x^*| \leq \sum_{i\in J(p)} E_i \, |\mathscr{L}^{(i)}(r,\omega)| \, |x^{s_i(p)} - x^*| + \sum_{i\notin J(p)} E_i \, |x^p - x^*| + \sum_{i\notin J(p)} E_i \, |R^{(i)}(x^{s_i(p)})|$$

$$\leq \sum_{i\in J(p)} E_i \, |\mathscr{L}^{(i)}(r,\omega)| \, \Delta_\iota x_\varepsilon + \sum_{i\notin J(p)} E_i \, |x^p - x^*| + \frac{\omega}{r}\varepsilon$$

$$\Delta_\iota \sum_{i\notin J(p)} E_i \, x_\varepsilon$$

$$\leq \sum_{i\in J(p)} E_i \, \sigma_\varepsilon \, \Delta_\iota x_\varepsilon + \sum_{i\notin J(p)} E_i \, |x^p - x^*|$$

$$= I_p^{(1)} \sigma_\varepsilon \, \Delta_\iota x_\varepsilon + I_p^{(2)} |x^p - x^*|$$

$$\leq \Big[I - \prod_{j=m_\iota}^{p} I_j^{(2)}\Big] \sigma_\varepsilon \, \Delta_\iota x_\varepsilon + \prod_{j=m_\iota}^{p} I_j^{(2)} |x^{m_\iota} - x^*|$$

$$\leq \Big[\sigma_\varepsilon I + (1-\sigma_\varepsilon) \prod_{j=m_\iota}^{p} I_j^{(2)}\Big] \Delta_\iota x_\varepsilon$$

$$\leq \Big[\sigma_\varepsilon I + (1-\sigma_\varepsilon) \, I^{(\iota+1)}\Big] \Delta_\iota x_\varepsilon$$

$$\leq \Big[\sigma_\varepsilon + (1-\sigma_\varepsilon) \, \gamma^{(\iota+1)}\Big] \Delta_\iota x_\varepsilon \, \Delta_0 = x_\varepsilon.$$

Now, According to the induction, (4.24) is proved.

Part III: Suppose $x^o \in N(x^*, \delta)$, then $\lim_{p\to\infty} x^p = x^*$

Let

$$\beta^{(\iota)} = \sigma_\varepsilon + (1-\sigma_\varepsilon) \, \gamma^{(\iota)}, \quad \iota = 0, 1, 2, \ldots$$

Clearly, $\{\beta^{(\iota)}\}_{\iota\in N_0} \subset [0,1)$. By the definition of $\{\Delta_\iota\}_{\iota\in N_0}$ we know that

$$\Delta_{\iota+1} = \beta^{(\iota+1)}, \quad \Delta_\iota = \prod_{J=0}^{\iota+1} \beta^{(j)} \, \delta \dashrightarrow 0 \quad (\iota \dashrightarrow \infty)$$

Take limits on both sides of (4.24), we immediately obtain

$$\lim_{p\to\infty} x^p = x^*$$

Up to now, the demonstration of theorem 1 is accomplished.

5. Convergence Analysis of Method-II

For each $i \in \{1, 2, \ldots, \alpha\}$, we define,

$$g_m^{(i)}(x) = \begin{cases} x_m - r \dfrac{f_m^{(i)}\left(x;\ \gamma^{m,i}(x)\right)}{H_{mm}^{(i)}\left(x;\ \gamma^{m,i}(x)\right)}, & \text{for } m \in J_i\ m = 1\,(1)n \\ x_m, & \text{for } m \notin J_i \end{cases} \tag{5.1}$$

with

$$\begin{cases} \gamma^{1,i}(x) = X \\ \gamma^{m,i}(x) = \left(g_1^{(i)}(x), \ldots, g_{m-1}^{(i)}(x), x_m, \ldots, x_n\right)^T & m = 1(1)n \end{cases} \tag{5.2}$$

and let

$$g^{(i)}(x) = \left(g_1^{(i)}(x), \ldots, g_{m-1}^{(i)}(x), x_m, \ldots, x_n\right)^T,$$

Method-II can also be expressed in the following equivalent form:

$$x^{p+1} = \sum_{i \in J(p)} E_i \left(\frac{\omega}{r} g^{(i)}\left(x^{s_i(p)}\right) + \left(1 - \frac{\omega}{r}\right) x^{s_i(p)}\right) + \sum_{i \notin J(p)} E_i x^p \tag{5.3}$$

Based on this, we can set up the local convergence theory about Method-II.

Theorem 2: Under the conditons of theorem 1, we additionally suppose that $H^{(i)}(x;y)$ is continuously differentiable in a neighbourhood of $(x^*;\ x^*)$ and satisfies

$$\lim_{(x;y) \to (x^*;x^*)} H_{mm}^{(i)}(x;y) = \partial_2^m f_m^{(i)}\left(x^*;\ x^*\right),\quad m = 1(1)n \tag{5.4}$$

for each $i \in \{1, 2, \ldots, \alpha\}$. Then, there exists a neighbourhood $N(x^*, \delta)$ of $x^* \in D$ such that the sequence $\{x^p\}_{p \in N_0}$ generated by Method-II starting from any initial approximation $x^0 \in N(x^*, \delta)$ converges to the solutoin x^* of the system of nonlinear equation (1.1).

Proof: Define ϱ_ε, σ_ε, J_ε and x_ε as (4.17)-(4.19), and we use the same notations as (4.1)-4.3) and (4.14). For sufficiently small $\tilde{\delta} > 0$, let $N(x^*, \tilde{\delta}) := \{x : |x - x^*| \leq \tilde{\delta}\, x_\varepsilon\} \subset D$ such that $f^{(i)}$ and $H^{(i)}$ are continuously differentiable on $N(x^*, \tilde{\delta}) \times N(x^*, \tilde{\delta})$ for each $i \in \{1, 2, \ldots, \alpha\}$

Write,

$$r_m^{(i)}(x;y) = f_m^{(i)}(x;y) - f_m^{(i)}(x^*,x^*) - (\partial_1 f_m^{(i)}(x^*;x^*)(x - x^*) + \partial_2^2 f_m^{(i)}(x^*,x^*)$$

$$(y - X^*)], \quad m = 1(1)n, \; i = 1, 2, \ldots, \alpha \tag{5.5}$$

making use of the induction, we can conclude that there exist $\delta^{(i)} \in (0, \tilde{\delta})$ and positive numbers $a_m^{(i)}$, $b_m^{(i)}$, $c_m^{(i)}$ $(m = 1\ (1)n.\ i = 1, 2, .., \alpha)$ such that

$$\begin{cases} \left| r_m^{(i)}(x;\ \gamma^{m,i}(x)) \right| \leq a_m^{(i)} \| x - x^* \|_\infty \\ \left| g_m^{(i)}(x) - g_m^{(i)}(x^*) \right| \leq b_m^{(i)} \| x - x^* \|_\infty, \quad \forall\, x \in N(x^*, \delta^{(i)}) \\ \| \gamma^{m,i}(x) - \gamma^{m,i}(x^*) \|_\infty \leq c_m^{(i)} \| x - x^* \| \end{cases} \tag{5.6}$$

hold for $m = 1(1)n$, $i = 1, 2, .., \alpha$.

In fact, from (5.1) - (5.2) we know that

$$g_m^{(i)}(x^*) = x^*_m, \text{ for } m = 1(1)n, \; i = 1, 2, .., \alpha. \tag{5.7}$$

and

$$g_m^{(i)}(x) - g_m^{(i)}(x^*) = x_m - x^*_m -$$

$$\begin{cases} \dfrac{r_m^{(i)}(x;\ \gamma^{m,i}(x)) + \partial_1 f_m^{(i)}(x^*;x^*)(x-x^*) + \partial_2 f_m^{(i)}(x^*;x^*)\ (\gamma^{m,i}(x)-x^*)}{H_{mm}^{(i)}(x;\ \gamma^{m,i}(x))}, & m \in J_i \\ 0, \quad m \notin J_i \end{cases} \tag{5.8}$$

Considering the continuous differentiability of $f_m^{(i)}$ and $H_{mm}^{(i)}$ $(m = 1(1)n;\ i = 1, 2, \ldots, \alpha)$ in $N(x^*, \tilde{\delta}) \times N(x^*, \tilde{\delta})$, by direct derivation we can get (5.6).

Then, there exists a neighbourhood N (x^*, δ) of $x^* \in D$ such that the sequence $\{x^P\}_{p \in No}$ generated by Mehod-I starting from any initial approximation $x^o \in N(x^*, \delta)$ converges to the solution x^* of the system of nonlinear equation (1.1) provided the relaxed parameters r and ω satisfy

$$o < r \leq \omega,\ 0 < \omega < 2 / (1 + \varrho(|D^{-1}|\ |B|)) \quad (4.5)$$

* * *

Proof: Because $x^* = (x_1, x_2, \ldots, x_n)^T \in D$ is a solution of system of nonlinear equation (1.1), there clearly have

$$f_m^{(i)}(x^*; x^*) = 0,\ m = 1(1)n,\ i = 1.\ 2, \ldots, \alpha.$$

Noticing

$$\partial_2^{(m)} f_m^{(i)}(x^*; x^*) \neq 0,\ m = 1(1)n,\ i = 1,\ 2, \ldots, \alpha.$$

in the light of the implicit function theorem, there exist two neighbourhoods $N(x^*, \delta_o)$, $N(x^*, \delta_1)$ of x^* such that for any $x \in N(x^*, \delta_o)$, the functions $T_m^{(i)}(x)$ $(m \in J_i,\ i = 1, 2, \ldots, \alpha)$ and

$$\tilde{y}^{(i)} : N(x^*, \delta_o) \dashrightarrow N(x^*, \delta_1),$$

$$\tilde{y}^{(i)}(x) = (\tilde{y}_1^{(i)}(x), \tilde{y}_2^{(i)}(x), \ldots, \tilde{y}_n^{(i)}(x))^T,\ i = 1,\ 2, \ldots, \alpha$$

are Successively determined by

$$f_m^{(i)}(x; \tilde{y}_1^{(i)}(x), \ldots, \tilde{y}_{m-1}^{(i)}(x), \tilde{y}_m^{(i)}(x), x_{m+1}, \ldots, x_n) = 0,\ m \in J_i,\quad i = 1, 2, \ldots, \alpha$$

and

$$\begin{cases} \tilde{y}_j^{(i)}(x) = \begin{cases} r\, T_j^{(i)}(x) + (1 - r)\, x_j, & \text{for } j \in J_i \\ x_j, & \text{for } j \notin J_i,\ i = 1, 2, \ldots, \alpha \end{cases} \\ j = 1, 2, \ldots, n \end{cases} \quad (4.6)$$

respectively, uniquely exist and are continuously differentiable in $N(x^*, \delta_o)$ with

$$\begin{cases} T_m^{(i)}(x^*) = x_m^*, & m \in J_i,\ i = 1, 2, \ldots, \alpha \\ y^{(i)}(x^*) = x^* \end{cases} \quad (4.7)$$

as well as

$$\begin{cases} - e_m^T L_i \dfrac{\partial \tilde{y}^{(i)}(x^*)}{\partial x_j} + b_{mm}^{(i)} \dfrac{\partial T_m^{(i)}(x^*)}{\partial x_j} - c_{mj}^{(i)} = \begin{cases} 0, & \text{for } j \le m \\ -b_{mj}^{(i)}, & \text{for } j > m \end{cases} \\ m, j = 1, 2, \ldots, n. \end{cases} \tag{4.8}$$

Making use of (4.6), we can get

$$\begin{cases} \dfrac{\partial \tilde{y}_m^{(i)}(x^*)}{\partial x_j} = \begin{cases} r \dfrac{\partial T_m^{(i)}(x^*)}{\partial x_j} + (1 - r), & \text{for } j = m \\ r \dfrac{\partial T_m^{(i)}(x^*)}{\partial x_j}, & \text{for } j \neq m \end{cases} \\ j = 1, 2, \ldots, n \quad i = 1, 2, \ldots, \alpha \end{cases} \tag{4.9}$$

Now, substitute (4.9) into (4.8), we obtain

$$\begin{cases} - r e_m^T L_i \dfrac{\partial \tilde{y}^{(i)}(x^*)}{\partial x_j} + b_{mm}^{(i)} \dfrac{\partial \tilde{y}^{(i)}_m(x^*)}{\partial x_j} - r c_{mj}^{(i)} = \begin{cases} 0, & \text{for } j < m \\ (1-r)\, b_{mm}^{(i)}, & \text{for } j=m \\ - r\, b_{mj}^{(i)}, & \text{for } j>m \end{cases} \\ j = 1, 2, \ldots, n\, ;\ i = 1, 2, \ldots, \alpha \end{cases}$$

or equivalently,

$$e_m^T \left((D_i - r L_i) \frac{\partial \tilde{y}^{(i)}(x^*)}{\partial x} - r (U_i + C_i) - (1 - r) D_i \right) = 0,$$

$$m \in J_i, \quad i = 1, 2, \ldots, \alpha. \tag{4.10}$$

Considering

$$\frac{\partial \tilde{y}_m^{(i)}(x^*)}{\partial x_j} = \begin{cases} 1, & \text{for } j = m,\ m \in J_i \\ & j = 1(1)n,\ i = 1, 2, \ldots, \alpha \\ 0, & \text{for } j \neq m,\ m \in J_i \end{cases} \tag{4.11}$$

we have

$$(D_i - r L_i \frac{\partial \tilde{y}^{(i)}(x^*)}{\partial x} = r (U_i + C_i) + (1 - r) D_i, \quad i = 1, 2, \ldots, \alpha.$$

Hence,

$$\frac{\partial \tilde{y}^{(i)}(x^*)}{\partial x} (D_i - r L_i)^{-1} \left[r (U_i + C_i) - (1 - r) D_i \right], \quad i = 1, 2, \ldots, \alpha \tag{4.12}$$

From (2.1) - (2.4) and (4.6), by direct manipulation, Method-I can be equivalently expressed as the following simple form:

$$x^{p+1} = \sum_{i \in J(p)} E_i \left(\frac{\omega}{r} \tilde{y}^{(i)} \left(x^{s_i(p)}\right) + \left(1 \frac{\omega}{r}\right) x^{s_i(p)}\right) + \sum_{i \notin J(p)} E_i x^p \tag{4.13}$$

Define:

$$\mathscr{L}^{(i)}(r, \omega) = (D_i - rL_i)^{-1} \left((1 - \omega) D_i + (\omega - r) L_i + \omega (U_i + C_i)\right),$$
$$i = 1, 2, \ldots, \alpha \tag{4.14}$$

and

$$R^{(i)}(x) = \frac{\omega}{r} \left(\tilde{y}^{(i)}(x) - \tilde{y}^{(i)}(x^*) - \frac{\partial \tilde{y}^{(i)}(x^*)}{\partial x} (x - x^*)\right), \ i = 1, 2, \ldots, \alpha \tag{4.15}$$

by using (4.12)-(4.13) we can conclude that

$$x^{p+1} - x^* = \sum_{i \notin J(p)} E_i \mathscr{L}^{(i)}(r, \omega) \left(x^{s_i(p)} - x^*\right) + \sum_{i \notin J(p)} E_i (x^p - x^*) + \sum_{i \notin J(p)} E_i R^{(i)}\left(x^{s_i(p)}\right) \tag{4.16}$$

Since $F'(x^*) \in L(R^n)$ is an H-matrix, $\varrho(|D|^{-1}|B|) < 1$. For any $\varepsilon > 0$, write

$$J_\varepsilon = |D|^{-1}|B| + \varepsilon e e^T, \ e = (1, 1, \ldots, 1)^T \in R^n \tag{4.17}$$

By continuity of the spectral radius of matrix and (4.5), we know that

$$\varrho_\varepsilon := \varrho(J_\varepsilon) < 1. \ \sigma_\varepsilon := \frac{\omega}{r} \varepsilon + |1 - \omega| + \omega \varrho_\varepsilon < 1 \tag{4.18}$$

provided ε is taken to be sufficiently small. Making use of Perron-Frobinus theorem in the nonnegative matrix theory, we see that there exists a positive vector $x_\varepsilon \in R^n$ satisfying

$$J_\varepsilon x_\varepsilon = \varrho_\varepsilon x_\varepsilon \tag{4.19}$$

For this ε, in accordance with the continuous differentiability of $\tilde{y}^{(i)}$: N (x^*, δ_0) -----> $N(x^*, \delta_1)$ for each $i \in \{1, 2, \ldots, \alpha\}$, there exists $\delta \in (0, \delta_0)$, such that for any $x \in N(x^*, \delta) := \{x : |x - x^*| \leq \delta x_\varepsilon\} \subset N(x^*, \delta_0)$, it holds that,

$$|\tilde{y}^{(i)}(x) - \tilde{y}^{(i)}(x^*) - \frac{\partial \tilde{y}^{(i)}(x^*)}{\partial x}(x - x^*)| \le \varepsilon |x - x^*|, \; i = 1, \; 2, \ldots, \alpha$$

Hence

$$|R^{(i)}(x)| \le \frac{\omega}{r} \varepsilon |x - x^*|, \; \forall \; x \in N(x^*, \delta), \; i = 1, 2, \ldots, \alpha \qquad (4.20)$$

can be concluded.

Noticing $(D_i - r L_i)$ $(i = 1, 2, \ldots, \alpha)$ are H-matrices, so

$$\begin{cases} |(D_i - r L_i)^{-1}| \le (|D_i| - r |L_i|)^{-1} \\ |(D_i| - r |L_i| \le (|D_i|, \; |D_i|^{-1} \le (|D_i| - r |L_i|)^{-1} \end{cases} \quad i = 1, 2, \ldots, \alpha \; (4.21)$$

Now, by making use of (4.2) - (4.5) and (4.17), from (4.14) we have

$$|\mathscr{L}^{(i)}(r, \omega)| \le |(D_i - r L_i|^{-1} \Big(|1 - \omega| \; |D_i| + (\omega - r) \; |L_i| + \omega \; |U_i + C_i|\Big)$$

$$\le (|D_i| - r \; |L_i|)^{-1} \Big(|1 - \omega| \; |D_i| + (\omega - r) \; |L_i|$$

$$+ \omega \; |U_i + C_i|\Big)$$

$$\le I + (|D_i| - r \; |L_i|)^{-1} \; |D_i| \Big((|1 - \omega| - 1) \; I + \omega \; |D|^{-1} \; |B|\Big)$$

$$\le I + (|D_i| - r \; |L_i|)^{-1} \; |D_i| \Big((|1 - \omega| - 1) \; I + \omega \; J_\varepsilon\Big),$$

$$i = 1, 2, \ldots, \alpha$$

Applying (4.18)-(4.19) and (4.21), we obtain

$$|\mathscr{L}^{(i)}(r, \omega)| \; x_\varepsilon \le (|1 - \omega| + \omega \, \varrho_\varepsilon) \; x_\varepsilon, \; i = 1, 2, \ldots, \alpha \qquad (4.22)$$

We continue to fulfill our proof within three parts.

Part I: Suppose $x^0 \in N(x^*, \delta)$, then $x^p \in N(x^*, \delta)$, $\forall \; p \in N_0$.

Obviously, $p = 0$ is a trivial case. Assume that for all $p \le t$, $x^p \in N(x^*, \delta)$ hold. Because of $s_i(t) \le t$ $(i = 1, 2, \ldots, \alpha)$,

$x^{s_i(t)} \in N(x^*, \delta)$ $(i = 1, 2, \ldots, \alpha)$. When $p = t + 1$, from (4.16), (4.18), (4.20) and (4.22), the following estimation can be conducted, The estimation (5.6) shows that $\gamma^{m,i}(x)$ and $g_m^{(i)}(x)$ are continuous in $N(x^*, \delta)$ for $m = 1(1)n$; $i = 1, 2, ..., \alpha$). With respect to each $i \in \{1, 2, ..., \alpha\}$, we now define the sets,

$$D_m^{(i)} = \left\{x \in N\left(x^*, \delta^{(i)}\right) : \partial_z^{(i)} f_m^{(i)}\left(x;\ \gamma^{m,i}(x)\right) \neq 0\right\},\ m = 1(1)n. \qquad (5.9)$$

The continuity of $\partial_2^{(i)} f_m^{(i)}$ in $N\left(x^*, \tilde{\delta}\right) \times N\left(x^*, \tilde{\delta}\right)$ and $\gamma^{m,i}$ in $N\left(x^*, \tilde{\delta}\right)$ as well as

$$\gamma^{m,i}(x^*) = x^*,\ \partial_z^{(i)} f_m^{(i)}\left(x^*;\ x^*\right) \neq 0,\ m = 1(1)n.$$

then implies that each $D_m^{(i)}$ is open. Using the continuous differentiability of $H^{(i)}$, again, we easily see that corresponding to each $D_m^{(i)}$, there exists a neighbourhood $D'^{(i)}_m \subset D_m^{(i)}$ of x^* such that

$$H_{mm}^{(i)}\left(x;\ \gamma^{m,i}(x)\right) \neq 0,\ \forall\ x \in D'^{(i)}_m$$

let

$$S_o^{(i)} = D'^{(i)}_1,\ S_m^{(i)} = S_{m-1}^{(i)} \cap D'^{(i)}_m,\ m = 1(1)n.$$

Obviously,

$$S_1^{(i)} = D'^{(i)}_1,\ S_m \subseteq S_{m-1}^{(i)},\ S_m^{(i)} \subseteq D'^{(i)}_m,\ m = 1(1)n.$$

By (5.1) (5.2), $g_m^{(i)}$ are well-defined in $S_m^{(i)}$ for $m = 1(1)n$. Since each $D'^{(i)}_m$ is open, so each $S_m^{(i)}$ is open, too. Take $\delta \in (0, \min_{1 \le i \le \alpha} \delta^{(i)})$, a neighbourhood

$$N(x^*, \delta) := \left\{x \ /\ \|x - x^*\| \le \delta\ x_\varepsilon\right\} \subset \bigcap_{i-1}^{\alpha} S_n^{(i)}$$

of x^* is therefore determined, which makes each $g^{(i)}$ ($i = 1, 2, .., \alpha$), and then Method-II is well-defined in $N(x^*, \alpha)$.

Write,

$$g^{(i)}(x) = \begin{cases} \left(H_{mm}^{(i)}\left(x;\ \gamma^{m,i}(x)\right) - \partial_2^{(m)} f_m^{(i)}\left(x^*;\ x^*\right)\right)\left(g_m^{(i)}(x) - g_m^{(i)}(x^*)\right. \\ \left.-\left(x_m - x^*_m\right)\right) + r\ f_m^{(i)}\left(x;\ \gamma^{m,i}(x)\right),\ m \in J_i, \\ -\ r\ e_m^T\ F'(x^*)\ (x - x^*),\ m \notin J_i,\ m = 1(1)n,\ i = 1, 2, ..., \alpha \end{cases} \qquad (5.10)$$

noticing (5.8) we can obtain

$$\begin{cases} \partial_2^{(m)} f_m^{(i)}(x^*; x^*)\left(g_m^{(i)}(x) - g_m^{(i)}(x^*)\right) = \partial_2^{(m)} f_m^{(i)}(x^*;x^*)\left(x_m - x_m^*\right) \\ \quad - r\left(\partial_1^{(m)} f_m^{(i)}(x^*; x^*)\left(x - x^*\right) + \partial_z^{(m)} f_m^{(i)}(x^*; x^*)\left(\gamma^{m,i}(x) - x^*\right)\right) - \\ \qquad q_m^{(i)}(x), \\ m \in J_i,\ m = 1\ (1)n,\ i = 1, 2, .., \alpha \end{cases}$$

i.e.,

$$\begin{cases} \partial_2^{(m)} f_m^{(i)}(x^*; x^*))\left(g_m^{(i)}(x) - g_m^{(i)}(x^*)\right) + \sum_{i=1}^{m-1} \partial_2^{(i)} f_m^{(i)}(x^*; x^*) \\ \qquad \left(g_j^{(i)}(x) - g_j^{(i)}(x^*)\right) \\ \partial_z^{(m)} f_m^{(i)}(x^*; x^*)\left(x - x^*\right) - r\,\partial_1^{(i)} f_m^{(i)}(x^*; x^*)\left(x - x^*\right) \\ + \sum_{i=1} \partial_z^{(i)} f_m^{(i)}(x^*; x^*)\left(x_j - x_j^*\right)) - q_m^{(i)}(x),\ m \in J_i, \\ \qquad m = 1(1)n,\ i = 1, 2, .., \alpha \end{cases} \tag{5.11}$$

By making use of (4.1) - (4.3) and (5.10), (5.11) can be equivalently expressed as

$$\begin{cases} \left(D_i - r L_i\right)\left(g^{(i)}(x) - g^{(i)}(x^*)\right) = \left((1 - r) D_i + r\left(U_i + C_i\right)\right)\left(x - x^*\right) \\ \qquad - q^{(i)}(x) \\ q^{(i)}(x) = q_1^{(i)}(x), q_2^{(i)}(x), \ldots, q_n^{(i)}(x))^T,\ i = 1, 2, .., \alpha \end{cases}$$

Thus, we have

$$g^{(i)}(x) - g^{(i)}(x^*) = (D_i - r L_i)^{-1}\left((1 - r) D_i + r (U_i + C_i)\right)\left(x - x^*\right) + R^{(i)}(x),\ i = 1, 2, .., \alpha \tag{5.12}$$

with

$$R^{(i)}(x) = -(D_i - r L_i)^{-1} q^{(i)}(x),\ i = 1, 2, .., \alpha \tag{5.13}$$

According to (5.6), there hold

$$|R^{(i)}(x)| \leq \varepsilon |x - x^*|,\ \forall\ x \in N(x^*, \delta),\ i = 1, 2, .., \alpha \tag{5.14}$$

for δ sufficiently small.

By using (5.3) and (5.12) - (5.13), we can conclude

$$x^{p+1} - x^* = \sum_{i \in J(p)} E_i \mathscr{L}^{(i)}(r, \omega) \left(x^{s_i(p)} - x^*\right) + \sum_{i \notin J(p)} E_i \left(x^p - x^*\right)$$

$$+ \sum_{i \notin J(p)} E_i \frac{\omega}{r} R^{(i)}\left(x^{s_i(p)}\right) \quad (5.15)$$

based on (5.14)- (5.15) similar to the proof of theorem 1, we can fulfull the proof of theorem 2, too.

We end this section with the following two remarks.

Remark I: The convergence theoiies of the asynchronous parallel nonlinear multisplitting AOR-Newton, -Chord, -Steffensen methods can be obtained as special cases of theorem 2.

Remark II: The varying intervals of the parameters r and ω in theorems 1 and 2 can be enlarged to

$$o \leq r \leq \omega, \; 0 < \omega < 2 / (1 + \varrho(|D|^{-1} |B|)).$$

References

1. Frommer, A. (1989) Parallel nonlinear multisplitting methods, Numer. Math., 56 : 269-282.
2. O'Leary, D.P., White, R.E. (1985) Multisplittings of matrices and parallel solution of linear systems, SIAM J. Alg. disc. Meth., 6 : 630-640.
3. White, R.E. (1986) A nonlinear parallel algorithm with application to the stefan problem, SIAM J. Numer. Anal., 23 : 639-652.
4. White, R.E. (1986) Parallel algorithm for nonlinear problems, SIAM J. Alg. Disc. Math., 7 : 137-149.
5. Wang Deren. (1991) On the convergence of the parallel multisplitting AOR algorithm, Linear Algebra Appl., 154-156 : 473-486.
6. Bru, R., Elsner, L. and Neumann, M. (1988) Models of parallel chaotic iteration methods, Linear Algebra Appl., 103 : 175-192.
7. Frommer, A. and Mayer, G. (1989) Convergence of relaxed parallel multisplitting methods, Linear Algebra Appl., 119 : 141-152.
8. Varga, R. S. (1962) Matrix Iterative Analysis, Prentice Hall, Englewood Cliffs, N.J.
9. Ortega, J.M. and Rheinboldt, W.C. (1990) Iterative Solutions of Nonlinear Equations in Several Variables, Academic Press, New York.
10. Evans, D.J. and Wang Deren. (1991) An asynchronous parallel algorithm for solving a class of nonlinear simultaneous equations, Parallel Comput., 17 : 165-180.