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A Theoretical Analysis of Simpson's 1/3-Type Scheme for the Riemann-Stieltjes Integral Based on Geometric Mean Derivative

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ABSTRACT

This research aims to discuss the theoretical analysis of a three-point numerical scheme used to approximate a Riemann-Stieltjes integral (RSI) by incorporating derivative in each strip at the geometric mean of the ending points of the interval of integration. The suggested scheme is discussed in terms of its basic and composite versions for the RSI. The error analysis of the scheme is also presented in theorems. Additionally, the reduction of the suggested scheme is discussed for the Riemann integral by using g(t) = t.

Keywords: Quadrature rule, Geometric mean, Riemann-Stieltjes, Error terms. Local error, Global error.

INTRODUCTION

The study of quadrature, or numerical integration, is a very vital and exciting area of research in numerical methods. In numerical integration, the primary goal is to study numerical methods that estimate the approximate value of a definite integral. The numerical value of a definite integral may be the area under a curve, surface area and volume of the solid figure, moment of inertia and many other important quantities. Unfortunately, some functions have no straightforward anti-derivatives for a definite integral in the case of non-linear arguments, for such circumstances, the value of a definite integral must be estimated numerically. Numerical methods are used to evaluate such definite integrals. Quadrature is the method of determining the approximate value of a defined integral. The Riemann-Stieltjes integral is a special case of the Riemann integral, and it is used in Stochastic and probabilistic methods, functions of complex variables and their integration, mathematical analysis of spaces including norms and inner products, theory of integral operators, etc. The RSI is defined in (Protter and Morrey, 1977) as:

$$RS(\mathrm{f}:\mathrm{g};\alpha,\beta)=\int_{\alpha}^{\beta} f(x)dg(x) \qquad (1)$$

Where f denotes the integrand and g the integrator.

Many authors have focused on and used quadrature rules to approximate the classical Riemann integral. The literature was given a new direction by the work of Zhao and Li (2013) who utilized midpoint derivatives to improve the precision and accuracy of the quadrature rules for approximating the Riemann integrals. A four-point scheme in Zhao and Li (2013) study was then improved computationally to include only second-order derivatives by Shaikh, Chandio and Soomro (2016) instead of fourth-order derivatives at midpoints without losing the accuracy. A series of works by a team of authors attempted to utilize the derivatives at other statistical means than just the arithmetic means of ending points of the interval of integration as proposed by Zhao and Li (2013). The works included the use of geometric (Ramachandran, Udyakumar and Parimala, 2016a), harmonic (Ramachandran, Udyakumar and Parimala, 2016b), heronian (Ramachandran, Udyakumar and Parimala, 2016c), centroidal (Ramachandran, Udyakumar and Parimala, 2016d) means; the degree of precision was enhanced by a unit than the classical derivative-free rules and the arithmetic mean methods had no restrictions on the choice of limits of integration (Ramachandran, Udyakumar and Parimala, 2016e). Bhatti et al., 2019 suggested combining lower order quadrature rules to attain higher accuracy based on the protocol which was suggested in the work of Amanat (2015). Khatri, Shaikh and Abro (2019) also worked on the modified quadrature methods with derivatives. Mahesar et al. (2022) utilized derivative-based improvements for integrals in the sense of semi-open rules as well as on open quadrature with heronian (Mahesar et al.; 2023a) and centroidal (Mahesar et al.; 2023b) means.

While a lot of researches were conducted for enhancement of the numerical schemes for the Riemann integral, only a few studies focused the numerical improvement of the RSIs. The pioneering work in this regard was of Mercer (2008) who worked on the trapezoidal rule approximation and consequent inequalities for the RSI. Later on in (Mercer, 2012), a similar discussion with relative convexity analysis was conducted for the open mid-point and closed Simpson's rules for the RSI. The derivative at midpoint of interval of integration was utilized by (Zhao, Zhang and Ye, 2014) in Trapezoid rule for the RSI only for the basic strip of integration, whereas the composite rules were discussed in (Zhao, Zhang and Ye, 2015). Recently, Memon et al. (2020a) worked on modified derivative-based rule of Trapezoid approximation for the RSI, whereas the four-point scheme which was derivative-free was also introduced in Memon et al. (2020b). The Simpson's type scheme for the RSI which was derive-free was examined in Memon et al. (2021a). Memon et al. (2021b) introduced derivative-based scheme for the RSI using geometric mean with three-point approximation. Besides, derivative-based improvements were also suggested for the higher dimensional integrals in the works of Malik et al. (2020a) and with error analyses in Malik et al. (2020b).

The focus of this study is analysis, extension and error terms of a recently introduced scheme of Simpson's type that utilized geometric mean derivatives for the RSI integral in Memon et al. (2021b). In this zeal, theorems concerning derivation, error analysis, reduction and extension to composite forms have been proved.

2. MAIN RESULTS

The basic form for approximating a Riemann integral with Simpson's 1/3 (S13) rule using derivative information at the geometric mean (GMS13) is defined in (2) as:

$$\int_{\alpha}^{\beta} f(x)dx = GMS13 = \left(\frac{\beta-\alpha}{6}\right)\left[f(\alpha)+4f\left(\frac{\alpha+\beta}{2}\right)+f(\beta)\right]-\frac{(\beta-\alpha)^5}{2880}f^{(4)}\left(\sqrt{\alpha\beta}\right)-\frac{(\beta-\alpha)^5\left(\sqrt{\beta}-\sqrt{\alpha}\right)^2}{5760}f^{(5)}(\xi), \quad (2)$$

Where $\xi \in (a,b)$. This rule has precision 4. On the basis of (2), in (Memon et al., 2021b), the proposed scheme, i.e. GMS13 for the RSI, in basic form was suggested. The proof of the derivation is discussed here as Theorem 1.

Theorem 1. Assuming that $f^{'}(t)$ and g (t) are continuous on $[\alpha,\beta]$ and $g(t)$ is also increasing on $[\alpha,\beta]$, then the basic form of the GMS13 can be described as follows:

$$\begin{aligned}\int_{\alpha}^{\beta} f(t)\,dg \approx GMS13 &= \left(\frac{4}{(\beta-\alpha)^2}I_2 - \frac{1}{\beta-\alpha}I_1 - g(\alpha)\right)f(\alpha)\\ &+\left(\frac{4}{\beta-\alpha}I_1 - \frac{8}{(\beta-\alpha)^2}I_2\right)f\left(\frac{\alpha+\beta}{2}\right)\\ &+\left(g(\beta) - \frac{3}{\beta-\alpha}I_1 + \frac{4}{(\beta-\alpha)^2}I_2\right)f(\beta)\\ &+\left(\frac{-(\beta-\alpha)^2(3\alpha+5\beta)}{96}I_1 + \frac{17\beta^2-10\alpha\beta-7\alpha^2}{48}I_2 - \beta I_3 + I_4\right)f^{(4)}\left(\sqrt{\alpha\beta}\right),\end{aligned} \tag{3}$$

Where

$$I_1 = \int_{\alpha}^{\beta} g(t)\,dt,\ I_2 = \int_{\alpha}^{\beta}\int_{\alpha}^{t} g(x)\,dxdt,\ I_3 = \int_{\alpha}^{\beta}\int_{\alpha}^{t}\int_{\alpha}^{y} g(x)\,dxdydt,\ I_4 = \int_{\alpha}^{\beta}\int_{\alpha}^{t}\int_{\alpha}^{z}\int_{\alpha}^{y} g(x)\,dxdydzdt$$

Proof of Theorem 1.

To proof the proposed Simpson's 1/3 - type scheme for the RSI, we need to find the numbers: a_0, b_0, c_0, d_0 to the following:

$$\int_{\alpha}^{\beta} f(t)\,dg \approx a_0 f(\alpha) + b_0 f\left(\frac{\alpha+\beta}{2}\right) + c_0 f(\beta) + d_0 f^{(4)}\left(\sqrt{\alpha\beta}\right) \tag{4}$$

that is exact for $f(t) = 1, t, t^2, t^3, t^4$. Therefore, we have

$$\int_{\alpha}^{\beta} 1\,dg = a_0 + b_0 + c_0$$

$$\int_{\alpha}^{\beta} t\,dg = a_0\alpha + b_0\left(\frac{\alpha+\beta}{2}\right) + c_0\beta$$

$$\int_{\alpha}^{\beta} t^2\,dg = a_0\alpha^2 + b_0\left(\frac{\alpha+\beta}{2}\right)^2 + c_0\beta^2$$

$$\int_{\alpha}^{\beta} t^3\,dg = a_0\alpha^3 + b_0\left(\frac{\alpha+\beta}{2}\right)^3 + c_0\beta^3$$

$$\int_{\alpha}^{\beta} t^4\,dg = a_0\alpha^4 + b_0\left(\frac{\alpha+\beta}{2}\right)^4 + c_0\beta^4 + 24d_0$$

We have applied the integration by parts technique of the RSI, as in (Zhao et al., 2014), we have got the following equations (5) - (9).

$$a_0 + b_0 + c_0 = g(\beta) - g(\alpha) \tag{5}$$

$$a_0\alpha + b_0\left(\frac{\alpha+\beta}{2}\right) + c_0\beta = \beta g(\beta) - \alpha g(\alpha) - I_1 \tag{6}$$

$$a_0\alpha^2 + b_0\left(\frac{\alpha+\beta}{2}\right)^2 + c_0\beta^2 = \beta^2 g(\beta) - \alpha^2 g(\alpha) - 2\beta I_1 + 2I_2 \tag{7}$$

$$a_0\alpha^3 + b_0\left(\frac{\alpha+\beta}{2}\right)^3 + c_0\beta^3 = \beta^3 g(\beta) - \alpha^3 g(\alpha) - 3\beta^2 I_1 + 6\beta I_2 - 6I_3 \tag{8}$$

$$a_0\alpha^4 + b_0\left(\frac{\alpha+\beta}{2}\right)^4 + c_0\beta^4 + 24d_0 = \beta^4 g(\beta) - \alpha^4 g(\alpha) - 4\beta^3 I_1 + 12\beta^2 I_2 - 24\beta I_3 + 24I_4 \tag{9}$$

Now we write the coefficient matrix to the above system of linear equations (5)-(9) as:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ \alpha & \dfrac{\alpha+\beta}{2} & \beta & 0 \\ \alpha^2 & \left(\dfrac{\alpha+\beta}{2}\right)^2 & \beta^2 & 0 \\ \alpha^3 & \left(\dfrac{\alpha+\beta}{2}\right)^3 & \beta^3 & 0 \\ \alpha^4 & \left(\dfrac{\alpha+\beta}{2}\right)^4 & \beta^4 & 24 \end{bmatrix}$$

The reduced row echelon form of M is:

$$M \overset{R}{\approx} M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since in M_R, $\operatorname{rank}(M) = 4$. Now we take $\alpha = -1$ and $\beta = 1$ in matrix M to check the linearly independent rows.

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 24 \end{bmatrix}$$

This indicates that all rows are linearly independent except the fourth row. Now we just solve the equations (5), (6), (7), and (9) simultaneously to find the coefficients a_0, b_0, c_0, and d_0.

$$a_0 = \frac{4}{(\beta-\alpha)^2} I_2 - \frac{1}{\beta-\alpha} I_1 - g(\alpha),$$

$$b_0 = \frac{4}{\beta-\alpha} I_1 - \frac{8}{(\beta-\alpha)^2} I_2,$$

$$c_0 = g(\beta) - \frac{3}{\beta-\alpha} I_1 + \frac{4}{(\beta-\alpha)^2} I_2,$$

$$d_0 = -\frac{(\alpha-\beta)^2(3\alpha+5\beta)}{96} I_1 + \frac{(17\beta^2 - 10\alpha\beta - 7\alpha^2)}{48} I_2 - \beta I_3 + I_4$$

Putting the values of coefficients a_0, b_0, c_0 and d_0 in (4), we have:

$$\begin{aligned}\int_{\alpha}^{\beta} f(t)\,dg \approx GMS13 = &\left(\frac{4}{(\beta-\alpha)^2}I_2 - \frac{1}{\beta-\alpha}I_1 - g(\alpha)\right) f(\alpha)\\ &+\left(\frac{4}{\beta-\alpha}I_1 - \frac{8}{(\beta-\alpha)^2}I_2\right) f\left(\frac{\alpha+\beta}{2}\right)\\ &+\left(g(\beta) - \frac{3}{\beta-\alpha}I_1 + \frac{4}{(\beta-\alpha)^2}I_2\right) f(\beta)\\ &+\left(\frac{-(\beta-\alpha)^2(3\alpha+5\beta)}{96}I_1 + \frac{17\beta^2-10\alpha\beta-7\alpha^2}{48}I_2 - \beta I_3 + I_4\right) f^{(4)}\left(\sqrt{\alpha\beta}\right)\end{aligned}$$

This is complete proof.

Let us derive the local error term of the proposed GS13 scheme in Theorem 2.

Theorem 2. Assuming that $f^{'}(t)$ and g (*t*) are continuous on $[\alpha,\beta]$ and $g(t)$ is also increasing on $[\alpha,\beta]$, then the local error term of RSI-GMS13 is obtained as:

$$\int_{\alpha}^{\beta} f(x)\,dg = GMS13 + R_{GMS13}[f]$$

with GMS13 discussed in (3), and the local error term $R_{GMS13}[f]$ described as:

$$\begin{aligned}R_{GMS13}[f] = &\left(\frac{-(\alpha-\beta)^2\left(7\alpha^2+16\alpha\beta-30\alpha\sqrt{\alpha\beta}-50\beta\sqrt{\alpha\beta}+17\beta^2\right)}{960}I_1\right.\\ &-\frac{(\alpha-\beta)(3\alpha^2+8\alpha\mu-14\alpha\sqrt{\alpha\beta}-34\beta\sqrt{\alpha\beta}+13\beta^2)}{96}I_2\\ &\left.+\frac{2\beta\sqrt{\alpha\beta}-\beta^2}{2}I_3 + \left(\beta-\sqrt{\alpha\beta}\right)I_4 - I_5\right) f^{(5)}(\xi)\,g'(\eta),\end{aligned} \qquad (10)$$

Where $I_5 = \int_{\alpha}^{\beta}\int_{\alpha}^{t}\int_{\alpha}^{w}\int_{\alpha}^{z}\int_{\alpha}^{y} g(x)\,dx\,dy\,dz\,dw\,dt$ and $\xi,\eta \in (\alpha,\beta)$.

Proof of Theorem 2.

The degree of precision of the GMS13 is 4, so the first non-zero error term which is also referred to as the leading error term in the approximation of RSI will be due to the polynomial $f(t)=\frac{t^5}{5!}$. The same is used to obtain the leading truncation error as:

$$R_{GMS13}[f] = \frac{1}{5!}\int_{\alpha}^{\beta} t^5\,dg - GMS13\left(t^5; g; \alpha, \beta\right)$$

(11) We learn, from (Zhao et al., 2014), that:

$$\frac{1}{5!}\int_{\alpha}^{\beta} t^5\,dg = \frac{1}{120}\left(\beta^5 g(\beta) - \alpha^5 g(\alpha)\right) - \frac{\beta^4}{24}I_1 + \frac{\beta^3}{6}I_2 - \frac{\beta^2}{2}I_3 + \beta I_4 - I_5 \qquad (12)$$

By Theorem 1 and scheme (3), we have:

$$\begin{aligned}
GMS13\left(t^5;g;a,\beta\right)=&\left(\frac{4}{(\beta-a)^2}I_2-\frac{1}{\beta-\alpha}I_1-g(\alpha)\right)\frac{\alpha^5}{5!}\\
&+\left(\frac{4}{\beta-\alpha}I_1-\frac{8}{(\beta-\alpha)^2}I_2\right)\frac{(\alpha+\beta)^5}{2^5.5!}\\
&+\left(g(\beta)-\frac{3}{\beta-a}I_1+\frac{4}{(\beta-\alpha)^2}I_2\right)\frac{\beta^5}{5!}\\
&+\left(\frac{-(\alpha-\beta)^2(3\alpha+5\beta)}{96}I_1+\frac{17\beta^2-10\alpha\beta-7\alpha^2}{48}I_2-\beta I_3+I_4\right)\sqrt{\alpha\beta}
\end{aligned}\tag{13}$$

We use (13) and (12) in (11) to get:

$$\begin{aligned}
R_{GMS13}[f]=&\Bigg(\frac{-(\alpha-\beta)^2\left(7\alpha^2+16\alpha\beta-30\alpha\sqrt{\alpha\beta}-50\beta\sqrt{\alpha\beta}+17\beta^2\right)}{960}I_1\\
&-\frac{(\alpha-\beta)(3\alpha^2+8\alpha\mu-14\alpha\sqrt{\alpha\beta}-34\beta\sqrt{\alpha\beta}+13\beta^2)}{96}I_2\\
&+\frac{2\beta\sqrt{\alpha\beta}-\beta^2}{2}I_3+\left(\beta-\sqrt{\alpha\beta}\right)I_4-I_5\Big)f^{(5)}(\xi)g'(\eta),
\end{aligned}\tag{14}$$

This the complete proof.

Theorem 3. If $g(t)=t$, then the proposed GMS13 scheme (10) for the RSI is equivalent to GMS13 scheme (Ramachandran, et al., 2016), i.e. (2) for the Riemann integrals.

Proof of Theorem 3. By Theorem 1, we have:

$$
\begin{aligned}
\int_{\alpha}^{\beta} f(t)\,dg = \int_{\alpha}^{\beta} f(t)dt = {} & \left(\frac{4}{(\beta-\alpha)^2}\int_{\alpha}^{\beta}\int_{\alpha}^{t} xdxdt - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} tdt - g(\alpha)\right) f(\alpha) \\
& + \left(\frac{4}{\beta-\alpha}\int_{\alpha}^{\beta} tdt - \frac{8}{(\beta-\alpha)^2}\int_{\alpha}^{\beta}\int_{\alpha}^{t} xdxdt\right) f\left(\frac{\alpha+\beta}{2}\right) \\
& + \left(g(\beta) - \frac{3}{\beta-\alpha}\int_{\alpha}^{\beta} tdt + \frac{4}{(\beta-\alpha)^2}\int_{\alpha}^{\beta}\int_{\alpha}^{t} xdxdt\right) f(\beta) \\
& + \left(\begin{array}{l} \frac{-(\beta-\alpha)^2(3\alpha+5\beta)}{96}\int_{\alpha}^{\beta} tdt + \frac{17\beta^2-10\alpha\beta-7\alpha^2}{48}\int_{\alpha}^{\beta}\int_{\alpha}^{t} xdxdt \\ -\beta\int_{\alpha}^{\beta}\int_{\alpha}^{t}\int_{\alpha}^{y} xdxdydt + \int_{\alpha}^{\beta}\int_{\alpha}^{t}\int_{\alpha}^{z}\int_{\alpha}^{y} xdxdydzdt \end{array}\right) f^{(4)}\left(\sqrt{\alpha\beta}\right) \\
& + \left(\frac{-(\alpha-\beta)^2\left(7\alpha^2+16\alpha\beta-30\alpha\sqrt{\alpha\beta}-50b\sqrt{\alpha\beta}+17\beta^2\right)}{960}\int_{\alpha}^{b} tdt\right. \\
& - \frac{(\alpha-\beta)(3\alpha^2+8\alpha\beta-14\alpha\sqrt{\alpha\beta}-34\beta\sqrt{\alpha\beta}+13\beta^2)}{96}\int_{\alpha}^{\beta}\int_{\alpha}^{t} xdxdt \\
& + \frac{2\beta\sqrt{\alpha\beta}-\beta^2}{2}\int_{\alpha}^{\beta}\int_{\alpha}^{t}\int_{\alpha}^{y} g(x)dxdydt + \left(\beta-\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\int_{\alpha}^{t}\int_{\alpha}^{z}\int_{\alpha}^{y} xdxdydzdt \\
& \left. - \int_{a}^{\beta}\int_{a}^{t}\int_{a}^{w}\int_{a}^{z}\int_{a}^{y} g(x)dxdydzdwdt\right) f^{(5)}(\xi)\,g'(\eta) \qquad (15)
\end{aligned}
$$

It is obvious to get:

$$\int_{\propto}^{\beta} tdt = \frac{\beta^2 - \propto^2}{2},$$

$$\int_{\propto}^{\beta}\int_{\propto}^{t} xdxdt = \frac{\beta^3}{6} - \frac{\propto^2\beta}{2} + \frac{\propto^3}{3},$$

$$\int_{\propto}^{\beta}\int_{\propto}^{t}\int_{\propto}^{y} xdxdy\,dt = \frac{\beta^4}{24} - \frac{\propto^2\beta^2}{4} + \frac{\propto^3\beta}{3} - \frac{\propto^4}{8},$$

$$\int_{\propto}^{\beta}\int_{\propto}^{t}\int_{\propto}^{z}\int_{\propto}^{y} x\,dxdydz\,dt = \frac{\beta^5}{120} - \frac{\propto^4\beta}{8} + \frac{\propto^3\beta^2}{6} - \frac{\propto^2\beta^3}{12} + \frac{\propto^5}{30},$$

$$\int_{\propto}^{\beta}\int_{\propto}^{t}\int_{\propto}^{w}\int_{\propto}^{z}\int_{\propto}^{y} xdxdydzdwdt = \frac{\beta^6}{720} + \frac{\propto^5\beta}{30} - \frac{\propto^4\beta^2}{16} + \frac{\propto^3\beta^3}{18} - \frac{\propto^2\beta^4}{48} - \frac{\propto^6}{144},$$

And, finally using these in (15), we get:

$$\int_{\alpha}^{\beta} f(x)dx = GMS13 = \left(\frac{\beta-\alpha}{6}\right)\left[f(\alpha) + 4f\left(\frac{\alpha+\beta}{2}\right) + f(\beta)\right] - \frac{(\beta-\alpha)^5}{2880} f^{(4)}\left(\sqrt{\alpha\beta}\right) - \frac{(\beta-\alpha)^5\left(\sqrt{\beta}-\sqrt{\alpha}\right)^2}{5760} f^{(5)}(\xi), \quad (16)$$

Where $\xi \in (\alpha, \beta)$.

This is complete proof of the reduction to the proposed GMS13 scheme for Riemann integral form as discussed in (2). Now we discuss the composite version of GMS13 scheme for the RSI that is obtained by splitting the interval into small sub-intervals. It is labelled GMCS13 and described in Theorem 4.

Theorem 4. Assuming that $f'(t)$ and g (t) are continuous on $[\alpha,\beta]$ and $g(t)$ is also increasing on $[\alpha,\beta]$ and that the interval $[\alpha,\beta]$ is split $[x_k,x_{k+1}]$ by the equally spaced nodes $x_k=\alpha+kh,$ into 2n subintervals using the width $h=\dfrac{\beta-\alpha}{n}$ and $k=0,1,...,n.$ Then, the composite GMS13 proposed scheme referred as GMCS13 for the RSI approximation with $2n$ subintervals for the RSI is:

$$\begin{aligned}\int_{\alpha}^{\beta} f(t)\,dg \approx GMCS13 = {} & \left[\frac{4n^2}{(\beta-\alpha)^2}\int_{\alpha}^{x_1}\int_{\alpha}^{t} g(x)dxdt-\frac{n}{\beta-\alpha}\int_{\alpha}^{x_1} g(t)dt-g(\alpha)\right]f(\alpha)\\
&+\frac{4n}{\beta-\alpha}\sum_{k=1}^{n}\left[\int_{x_{k-1}}^{x_k} g(t)dt-\frac{2n}{\beta-\alpha}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)\\
&+\frac{n}{\beta-\alpha}\sum_{k=1}^{n-1}\left[\frac{4n}{\beta-\alpha}\left(\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} g(x)dxdt\right)-\left(3\int_{x_{k-1}}^{x_k} g(t)dt+\int_{x_k}^{x_{k+1}} g(t)dt\right)\right]f\left(x_k\right)\\
&+\sum_{k=1}^{n}\left[\begin{matrix}\dfrac{-h^2}{96}\left(3x_{k-1}+5x_k\right)\displaystyle\int_{x_{k-1}}^{x_k} g(t)dt+\dfrac{17x_k^2-10x_{k-1}x_k-7x_{k-1}^2}{48}\displaystyle\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\\
-x_k\displaystyle\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y} g(x)dxdydt+\displaystyle\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y} g(x)dxdydzdt\end{matrix}\right]f^{(4)}\left(\sqrt{x_{k-1}x_k}\right)\\
&+\left[g(\beta)-\frac{3n}{\beta-\alpha}\int_{x_{n-1}}^{\beta} g(t)dt+\frac{4n^2}{(\beta-\alpha)^2}\int_{x_{n-1}}^{\beta}\int_{x_{n-1}}^{t} g(x)dxdt\right]f(\beta) \qquad (17)\end{aligned}$$

Proof of Theorem 4.

The proposed basic GMS13 scheme for the RSI is given in (3). We have proposed GMS13 rule over each subinterval, we have

$$\begin{aligned}\int_{\alpha}^{\beta} f(t)\,dg \approx {} & \left[\frac{4}{\left(\dfrac{\beta-\alpha}{n}\right)^2}\int_{\alpha}^{x_1}\int_{\alpha}^{t} g(x)dxdt-\frac{1}{\dfrac{\beta-\alpha}{n}}\int_{\alpha}^{x_1} g(t)dt-g(\alpha)\right]f\left(\alpha\right)\\
&+\left[\frac{4}{\dfrac{\beta-\alpha}{n}}\int_{\alpha}^{x_1} g(t)dt-\frac{8}{\left(\dfrac{\beta-\alpha}{n}\right)^2}\int_{\alpha}^{x_1}\int_{\alpha}^{t} g(x)dxdt\right]f\left(\frac{\alpha+x_1}{2}\right)\\
&+\left[g(x_1)-\frac{3}{\dfrac{\beta-\alpha}{n}}\int_{\alpha}^{x_1} g(t)dt+\frac{4}{\left(\dfrac{\beta-\alpha}{n}\right)^2}\int_{\alpha}^{x_1}\int_{\alpha}^{t} g(x)dxdt\right]f\left(x_1\right)\end{aligned}$$

$$+\left(\begin{array}{l}\dfrac{-\left(\dfrac{\beta-\alpha}{n}\right)^{2}\left(3\alpha+5x_1\right)}{96}\displaystyle\int_{\alpha}^{x_1} g(t)dt+\dfrac{17x_1^2-10\alpha x_1-7\alpha^2}{48}\int_{\alpha}^{x_1}\int_{\alpha}^{t} g(x)dxdt\\ -x_1\displaystyle\int_{\alpha}^{x_1}\int_{\alpha}^{t}\int_{\alpha}^{y} g(x)dxdydt+\int_{\alpha}^{x_1}\int_{\alpha}^{t}\int_{\alpha}^{z}\int_{\alpha}^{y} g(x)dxdydzdt\end{array}\right)f^{(4)}\left(\sqrt{\alpha x_1}\right)$$

$$+\left[\frac{4}{\left(\dfrac{\beta-\alpha}{n}\right)^{2}}\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt-\frac{1}{\dfrac{\beta-\alpha}{n}}\int_{x_1}^{x_2} g(t)dt-g(x_1)\right]f(x_1)$$

$$+\left[g(x_2)-\frac{3}{\dfrac{\beta-\alpha}{n}}\int_{x_1}^{x_2} g(t)dt+\frac{4}{\left(\dfrac{\beta-\alpha}{n}\right)^{2}}\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\right]f\left(x_2\right)$$

$$+\left[\frac{4}{\dfrac{\beta-\alpha}{n}}\int_{x_1}^{x_2} g(t)dt-\frac{8}{\left(\dfrac{\beta-\alpha}{n}\right)^{2}}\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\left[g(x_2)-\frac{3}{\dfrac{\beta-\alpha}{n}}\int_{x_1}^{x_2} g(t)dt+\frac{4}{\left(\dfrac{\beta-\alpha}{n}\right)^{2}}\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\right]f\left(x_2\right)$$

$$+\left(\begin{array}{l}\dfrac{-\left(\dfrac{\beta-\alpha}{n}\right)^{2}\left(3x_1+5x_2\right)}{96}\displaystyle\int_{x_1}^{x_2} g(t)dt+\dfrac{17x_2^2-10x_1x_2-7x_1^2}{48}\int_{x_1}^{x_2}\int_{x_1}^{t} g(x)dxdt\\ -x_2\displaystyle\int_{x_1}^{x_2}\int_{x_1}^{t}\int_{x_1}^{y} g(x)dxdydt+\int_{x_1}^{x_2}\int_{x_1}^{t}\int_{x_1}^{z}\int_{x_1}^{y} g(x)dxdydzdt\end{array}\right)f^{(4)}\left(\sqrt{x_1x_2}\right)$$

$$+\ldots+\left[\frac{4}{\left(\dfrac{\beta-\alpha}{n}\right)^{2}}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt-\frac{1}{\dfrac{\beta-\alpha}{n}}\int_{x_{k-1}}^{x_k} g(t)dt-g(x_{k-1})\right]f\left(x_{k-1}\right)$$

$$+\left[\frac{4}{\dfrac{\beta-\alpha}{n}}\int_{x_{k-1}}^{x_k} g(t)dt-\frac{8}{\left(\dfrac{\beta-\alpha}{n}\right)^{2}}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\left[g(x_k)-\frac{3}{\frac{\beta-\alpha}{n}}\int_{x_{k-1}}^{x_k}g(t)dt+\frac{4}{\left(\frac{\beta-\alpha}{n}\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f\left(x_k\right)$$

$$+\left(\begin{array}{l}\dfrac{-\left(\frac{\beta-\alpha}{n}\right)^2\left(3x_{k-1}+5x_k\right)}{96}\displaystyle\int_{x_{k-1}}^{x_k}g(t)dt+\dfrac{17x_k^2-10x_{k-1}x_k-7x_{k-1}^2}{48}\displaystyle\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\\ -x_k\displaystyle\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{y}g(x)dxdydt+\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}\int_{x_{k-1}}^{z}\int_{x_{k-1}}^{y}g(x)dxdydzdt\end{array}\right)f^{(4)}\left(\sqrt{x_{k-1}x_k}\right)$$

$$+\ldots+\left[\frac{4}{\left(\frac{\beta-\alpha}{n}\right)^2}\int_{x_{n-1}}^{\beta}\int_{x_{n-1}}^{t}g(x)dxdt-\frac{1}{\frac{\beta-\alpha}{n}}\int_{x_{n-1}}^{\beta}g(t)dt-g(x_{n-1})\right]f\left(x_{n-1}\right)$$

$$+\left[\frac{4}{\frac{\beta-\alpha}{n}}\int_{x_{n-1}}^{\beta}g(t)dt-\frac{8}{\left(\frac{\beta-\alpha}{n}\right)^2}\int_{x_{n-1}}^{\beta}\int_{x_{n-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{n-1}+\beta}{2}\right)$$

$$+\left[g(\beta)-\frac{3}{\frac{\beta-\alpha}{n}}\int_{x_{n-1}}^{\beta}g(t)dt+\frac{4}{\left(\frac{\beta-\alpha}{n}\right)^2}\int_{x_{n-1}}^{\beta}\int_{x_{n-1}}^{t}g(x)dxdt\right]f\left(\beta\right)$$

$$+\left(\begin{array}{l}\dfrac{-\left(\frac{\beta-\alpha}{n}\right)^2\left(3x_{n-1}+5\beta\right)}{96}\displaystyle\int_{x_{n-1}}^{\beta}g(t)dt+\dfrac{17\beta^2-10x_{n-1}\beta-7x_{n-1}^2}{48}\displaystyle\int_{x_{n-1}}^{\beta}\int_{x_{n-1}}^{t}g(x)dxdt\\ -\beta\displaystyle\int_{x_{n-1}}^{b}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{y}g(x)dxdydt+\int_{x_{n-1}}^{\beta}\int_{x_{n-1}}^{t}\int_{x_{n-1}}^{z}\int_{x_{n-1}}^{y}g(x)dxdydzdt\end{array}\right)f^{(4)}\left(\sqrt{x_{n-1}\beta}\right)$$

$$=\left[\frac{4n^2}{\left(\beta-\alpha\right)^2}\int_{\alpha}^{x_1}\int_{\alpha}^{t}g(x)dxdt-\frac{n}{\beta-\alpha}\int_{\alpha}^{x_1}g(t)dt-g(\alpha)\right]f\left(\alpha\right)$$

$$+\left[\frac{4n}{\beta-\alpha}\int_{\alpha}^{x_1}g(t)dt-\frac{8n^2}{\left(\beta-\alpha\right)^2}\int_{\alpha}^{x_1}\int_{\alpha}^{t}g(x)dxdt\right]f\left(\frac{\alpha+x_1}{2}\right)$$

$$+\left[\frac{4n}{\beta-\alpha}\int_{x_1}^{x_2}g(t)dt-\frac{8n^2}{\left(\beta-\alpha\right)^2}\int_{x_1}^{x_2}\int_{x_1}^{t}g(x)dxdt\right]f\left(\frac{x_1+x_2}{2}\right)$$

$$+\ldots+\left[\frac{4n}{\beta-\alpha}\int_{x_{k-1}}^{x_k}g(t)dt-\frac{8n^2}{\left(\beta-\alpha\right)^2}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{t}g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\dots+\left[\frac{4n}{\beta-\alpha}\int\limits_{x_{n-1}}^{\beta} g(t)dt-\frac{8n^2}{(\beta-\alpha)^2}\int\limits_{x_{n-1}}^{\beta}\int\limits_{x_{n-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{n-1}+\beta}{2}\right)$$

$$+\left[\frac{4n^2}{(\beta-\alpha)^2}\left(\int\limits_{\alpha}^{x_1}\int\limits_{\alpha}^{t} g(x)dxdt+\int\limits_{x_1}^{x_2}\int\limits_{x_1}^{t} g(x)dxdt\right)-\frac{n}{\beta-\alpha}\left(3\int\limits_{a}^{x_1} g(t)dt+\int\limits_{x_1}^{x_2} g(t)dt\right)\right]f\left(x_1\right)$$

$$+\left[\frac{4n^2}{(\beta-\alpha)^2}\left(\int\limits_{x_1}^{x_2}\int\limits_{x_1}^{t} g(x)dxdt+\int\limits_{x_2}^{x_3}\int\limits_{x_2}^{t} g(x)dxdt\right)-\frac{n}{\beta-\alpha}\left(3\int\limits_{x_1}^{x_2} g(t)dt+\int\limits_{x_2}^{x_3} g(t)dt\right)\right]f\left(x_2\right)$$

$$+\dots+\left[\frac{4n^2}{(\beta-\alpha)^2}\left(\int\limits_{x_{n-2}}^{x_{n-1}}\int\limits_{x_{n-2}}^{t} g(x)dxdt+\int\limits_{x_{n-1}}^{x_n}\int\limits_{x_{n-1}}^{t} g(x)dxdt\right)-\frac{n}{\beta-\alpha}\left(3\int\limits_{x_{n-2}}^{x_{n-1}} g(t)dt+\int\limits_{x_{n-1}}^{x_n} g(t)dt\right)\right]f\left(x_{n-1}\right)$$

$$+\left[g(\beta)-\frac{3n}{\beta-\alpha}\int\limits_{x_{n-1}}^{\beta} g(t)dt+\frac{4n^2}{(\beta-\alpha)^2}\int\limits_{x_{n-1}}^{\beta}\int\limits_{x_{n-1}}^{t} g(x)dxdt\right]f\left(\beta\right)$$

$$+\left(\begin{array}{l}\dfrac{-h^2\left(3x_{n-1}+5\beta\right)}{96}\displaystyle\int\limits_{x_{n-1}}^{\beta} g(t)dt+\dfrac{17\beta^2-10x_{n-1}\beta-7x_{n-1}^2}{48}\displaystyle\int\limits_{x_{n-1}}^{\beta}\int\limits_{x_{n-1}}^{t} g(x)dxdt\\ -\beta\displaystyle\int\limits_{x_{n-1}}^{\beta}\int\limits_{x_{n-1}}^{t}\int\limits_{x_{n-1}}^{y} g(x)dxdydt+\int\limits_{x_{n-1}}^{\beta}\int\limits_{x_{n-1}}^{t}\int\limits_{x_{n-1}}^{z}\int\limits_{x_{n-1}}^{y} g(x)dxdydzdt\end{array}\right)f^{(4)}\left(\sqrt{x_{n-1}\beta}\right)$$

$$=\left[\frac{4n^2}{(\beta-\alpha)^2}\int\limits_{\alpha}^{x_1}\int\limits_{\alpha}^{t} g(x)dxdt-\frac{n}{\beta-\alpha}\int\limits_{\alpha}^{x_1} g(t)dt-g(\alpha)\right]f\left(\alpha\right)$$

$$+\frac{4n}{\beta-\alpha}\sum_{k=1}^{n}\left[\int\limits_{x_{k-1}}^{x_k} g(t)dt-\frac{2n}{\beta-\alpha}\int\limits_{x_{k-1}}^{x_k}\int\limits_{x_{k-1}}^{t} g(x)dxdt\right]f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$+\frac{n}{\beta-\alpha}\sum_{k=1}^{n-1}\left[\frac{4n}{\beta-\alpha}\left(\int\limits_{x_{k-1}}^{x_k}\int\limits_{x_{k-1}}^{t} g(x)dxdt+\int\limits_{x_k}^{x_{k+1}}\int\limits_{x_k}^{t} g(x)dxdt\right)-\left(3\int\limits_{x_{k-1}}^{x_k} g(t)dt+\int\limits_{x_k}^{x_{k+1}} g(t)dt\right)\right]f\left(x_k\right)$$

$$+\left[g(\beta)-\frac{3n}{\beta-\alpha}\int\limits_{x_{n-1}}^{\beta} g(t)dt+\frac{4n^2}{(\beta-\alpha)^2}\int\limits_{x_{n-1}}^{\beta}\int\limits_{x_{n-1}}^{t} g(x)dxdt\right]f\left(\beta\right)$$

$$+\sum_{k=1}^{n}\left(\begin{array}{l}\dfrac{-h^2\left(3x_{k-1}+5x_k\right)}{96}\displaystyle\int\limits_{x_{k-1}}^{x_k} g(t)dt+\dfrac{17x_k^2-10x_{k-1}x_k-7x_{k-1}^2}{48}\displaystyle\int\limits_{x_{k-1}}^{x_k}\int\limits_{x_{k-1}}^{t} g(x)dxdt\\ -x_k\displaystyle\int\limits_{x_{k-1}}^{x_k}\int\limits_{x_{k-1}}^{t}\int\limits_{x_{k-1}}^{y} g(x)dxdydt+\int\limits_{x_{k-1}}^{x_k}\int\limits_{x_{k-1}}^{t}\int\limits_{x_{k-1}}^{z}\int\limits_{x_{k-1}}^{y} g(x)dxdydzdt\end{array}\right)f^{(4)}\left(\sqrt{x_{k-1}\beta}\right)$$

This is complete proof.

The global error term of the proposed GMCS13 scheme is not described in classical form due to the involvement of non-standard term $\left(\sqrt{\beta}-\sqrt{\alpha}\right)$ besides the only required standard powers of $\left(\beta-\alpha\right)$.

3. RESULTS AND DISCUSSION

The theoretical analysis presented in section 2 focused on the derivation of GMS13 to approximate RSI was discussed with error term. The consequent extension for the composite application and related theorems were also proved. All the theoretical results match the claimed numerical properties in the study of Memon et al. (2021b) with regard to the error distributions, computational overhead and execution times (in sec.). This confirms the validation of the results derived.

4. CONCLUSION

An efficient approximation of the RSI was focused in this study by considering Simpson's rule with three points and the derivative content at the geometric mean. The theorems on basic forms of the scheme, local error term, composite form and reduction were proved successfully. The discussed scheme exhibits enhanced degree of precision and accuracy than those reported for the conventional and existing schemes: Trapezoid derivative-free, Simpson's derivative-free, etc.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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