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Contra-Harmonic Mean Derivative-Based Open Newton-Cotes Quadrature Rules

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ABSTRACT

This research focuses upon the numerical assessment of definite integrals by developing a family of derivative-based numerical integrators, which utilize the even-order (second and fourth- order) derivatives at the contra-harmonic-mean of the boundary points of the interval of integration. In comparison to well-known closed-form integrators by Newton and Cotes, the new methods improve accuracy by two orders of magnitude. These formulas arise from the concept of degree of precision. Furthermore, the theoretical conclusions are validated by calculating the computational order of accuracy for each approach. The proposed class of numerical integrators exhibit superior and efficient performance when compared to the closed-form rules by Newton and Cotes in terms of both the error drops and computational overhead.

Keywords: Quadrature rule, Contra-Harmonic mean, Open Newton-Cotes, Error terms, Local error, Global error.

INTRODUCTION

A vital part of numerical analysis that is necessary for resolving numerous real-world issues is numerical integration. Numerous methods have been created over time to successfully handle a range of integration difficulties. These techniques are generally relevant in domains where numerical computation of integrals is commonly required, such as physics and engineering (Shaikh, 2019). The fundamental method entails using simple integration principles and splitting the integration interval into equally spaced subintervals. A general quadrature formula for achieving numerical integration is described in (1).

$$\int_a^b g(x) \approx \sum_{i=0}^{n} c_i g(x_i) \tag{1}$$

Equation (1) shows $(n+2)$ intermediate locations so that the interval $a < x < b$ is sub-divided into finite number of subintervals $a < x_0 < x_1 < x_2 < \cdots < x_n < b$; where $x_i = a + i\eta$; $i = 1,2,3 \ldots n-1$; $\eta = (b-a)/(n+2)$ and $c_i's$ are the weights. The values of the weights c_i, for $i = 0,1,2, \ldots, n$ are to be chosen such that the techniques based on precision have no error, as defined in (2).

$$E_n = \int_a^b f(x) - \sum_{i=0}^{n} c_i g(x_i) \qquad (2)$$

Closed and open formulas are the two types of Newton-Cotes rules. In closed formulas (Shaikh, Chandio and Soomro, 2016), the closed intervals are included in the integration, while open intervals are applied in open formulas. Open Newton-Cotes (ONC) quadrature formula commonly known method for checking numerical integration, particularly in cases involving nonlinearities and singularities. Some open formulas that are based on whether n is even or odd are given as:

Midpoint Rule for $n = 0$

$$\int_a^b g(x)dx = (b-a)\left[g\left(\frac{a+b}{2}\right)\right] + \frac{(b-a)^3}{24} g''(\xi) \qquad (3)$$

Where, $\xi \epsilon (a,b)$. This method has the precision of degree one and achieves third-order accuracy.

One-point Rule for $n = 1$

$$\int_a^b g(x)dx = \frac{b-a}{2}\left[g\left(\frac{2a+b}{3}\right) + g\left(\frac{a+2b}{3}\right)\right] + \frac{(b-a)^3}{36} g''(\xi) \qquad (4)$$

Where, $\xi \epsilon (a,b)$. This method has the precision of degree one and achieves third-order accuracy.

Two-Point Rule for $n = 2$

$$\int_a^b g(x)dx = \frac{b-a}{3}\left[2g\left(\frac{3a+b}{4}\right) - g\left(\frac{a+b}{2}\right) + 2g\left(\frac{a+3b}{4}\right)\right] + \frac{7(b-a)^5}{23040} g^{(4)}(\xi) \qquad (5)$$

Where $\xi \epsilon (a,b)$. This method has the precision of degree three and achieves fifth-order accuracy.

Various strategies have been developed over time to refine and increase the accuracy of the traditional Newton-Cotes integration formulas. Dehghan et al. (2005a) improved the Closed Newton-Cotes rule (CNC) through the enhancement two or more parameters while the exchanging the interval's borders and developing the exact integral to reach the desired boundary point. Dehghan et al. (2006; 2005b) expanded the work using this technique to obtain the open and semi-open Newton-Cotes rules, Gauss-Legendre quadrature formulas by Babolian et al. (2005) and Gauss-Chebyshev quadrature rules by Eslahchi et al. (2004). Newton-Cotes quadrature rules are obtained by replacing the integrand with an interpolating polynomial of the appropriate degree. The technique of applying intermediate coefficients was used by Sermutlua (2019) to enhance these rules. Compared to conventional rules, these formulas are more accurate and have enhanced precision. It was discovered that a number of new quadrature rules meet the greatest efficiency qualities for closed-form and open-type quadrature by Newton and Cotes for a certain class of weighting functions (Kavitha and Thirumalai, 2020). The Newton-Cotes closed and open quadrature formulas are comparable. Simos (2008) discussed extremely interesting application of the trigonometrically fitted symplectic techniques based on the CNC formulas for the integration of a few physical problems.

By employing the sextic B-spline collocation approach, Nasir et al. (2023) approximated the solution to the generalized equal width wave equation. The computer results showed that the new findings were superior than those from earlier studies. Turkyilmazoglu (2014a) attempted to approximate the numerical solution of higher order mathematical equations involving derivative and integral of the unknown function using easy-to-understand protocol. The same protocol was also extended to numerically solve the higher order mixed differential-integral equations of the Volterra, Fredholm and Hammerstein type. The improved method solved nonlinear algebraic equations rather than linear ones. The accuracy of their proposed method for nonlinear equations was higher than that of several of the current methods. A technique for achieving the analytic approximation solution of the nonlinear differential equations arising from heat transport problems, namely through fins, was devised by (Turkyilmazoglu, 2014b).

Burg (2012; 2013) presented new closed quadrature rules and midpoint-based quadrature techniques together with corresponding theoretical findings. This method made use of the derivatives that were assessed at different times during the integration period. Zafar et al. (2014) built on this basis by developing derivative-based open Newton-Cotes (ONC) formulas, which improved precision by two orders of magnitude over the conventional closed Newton-Cotes (CNC) rules. The midpoint was then defined as the arithmetic mean of the interval's endpoints. Zhao and Li (2014) developed a derivative-based trapezoidal rule specifically designed for the Riemann-Stieltjes integral in their subsequent work. Memon et al. (2020) also suggested derivative-driven approaches, such as a specific four-point integration method proposed by Memon et al. (2021) for the Riemann-Stieltjes integration. Memon et al. (2025) suggested novel trapezoid-type quadrature rules for approximating the Riemann–Stieltjes integral that were both economical and time-efficient. The new formulas maintained good accuracy while drastically reducing computational effort. Their superior performance over current trapezoid-based techniques was demonstrated by numerical tests. The effectiveness of a four-point Simpson's quadrature rule for calculating Riemann-Stieltjes integrals was assessed by Memon et al. (2025). Through a number of numerical experiments, the authors examined accuracy, convergence behavior, and error performance. The suggested approach was effective and provided better precision than conventional quadrature systems, according to the results. Shaikh et al. (2025) explored the correctness of Simpson’s 1/3-type quadrature formula for the Riemann–Stieltjes integral using a geometric-mean-based derivative. In comparison to traditional Simpson-type rules, the authors show enhanced precision and develop theoretical error boundaries. The technique was shown to be an effective substitute for integrals employing non-smooth integrators in the study. These developments in integration techniques can be used in a variety of fields, including functional analysis (Pughachev et al, 1999)), stochastic processes (Roseburg (1985)) and Probabilty theory (Billingsley, P. 2017). In addition to standard quadrature formulas, researchers have introduced a number of novel methods for assessing integrals by combining derivatives of the integrand at different statistical means. For algebraic functions in particular, (Ramachandran and Parimala, 2015) established open Newton-Cotes (ONC) rule that makes use of the function's derivative at the midpoint of the integration interval. In further developments, Furthermore, a number of other approaches to CNC rules have been put out, combining derivatives with the original function. Techniques based on harmonic mean and contra-harmonic mean (Rana, 2022) and centroidal mean (Ramachandran and Parimala, 2016) are among them. Particularly in terms of reducing error and improving the quality of integral approximations, derivative-based CNC schemes based on these statistical approaches have proven to be more accurate than the traditional CNC formulas.

Mahesar et al. (2023a; 2023b; 2025) proposed the techniques for open Newton-Cotes (ONC) formulas where the Heronian and other means of the interval's ends were used in the derivatives of the function.

A collection of inequalities were studied by Demir and Sanal (2023) that apply to functions that are strong and n-times differentiable s-convexity. These inequalities were constructed using a perturbed variant of the trapezoidal inequality. Tighter boundaries were shown by their findings when applied to highly functions that are s-convex. Additionally, when the constant linked to strong. The obtained theorems readily reduce to those that apply to standard assumptions when s-convexity is set to zero. functions that are s-convex.

Mahesar et al. (2022; 2025) have developed new derivative-based semi-open quadrature algorithms to efficiently estimate definite integrals. By carefully evaluating errors, the authors demonstrate improved accuracy and computational performance when compared to conventional Newton–Cotes formulas. Numerical experiments verify the effectiveness and reliability of the proposed methods.

While there is a lot of potential for further investigation, the open Newton-Cotes (ONC) rules have received less attention in the literature than the closed Newton-Cotes (CNC) formulas (Malik et al., 2020a; 2020b). Developing new derivative-based ONC schemes using a variety of statistical techniques is one encouraging possibility. In this regard, the current study presents a family of effective ONC techniques in which the Contra-Harmonic mean assessed at the endpoints of the integration interval $[a, b]$ is used to incorporate the function's derivatives in addition to its values. The Contra-Harmonic mean derivative-based open Newton-Cotes (CHMDONC) rule is the name given to this innovative method. It has been shown that the suggested formulation offers higher computational efficiency and improves the accuracy and precision of traditional ONC approaches by two orders of magnitude. Additionally, computer analyses of error terms are carried out to evaluate the accuracy of suggested method with existing methods.

2. MATERIALS AND METHODS

For open Newton-Cotes rules, the general quadrature rule is represented as follows:

$$\int_{a}^{b} g(x)dx \approx \int_{x_0}^{x_1} f(x) \approx \sum_{i=1}^{n-1} q_i g(x_i) \qquad (6)$$

In this formulation of end points of the interval $[x_0, x_1]$, are removed from the quadrature rule where q_i are $n-1$ weights and x_i are $n-1$ distinct point within the interval and $h = \frac{b-a}{n+2}$; $x_i = x_0 + ih$. Hence, the composite forms of open Newton-Cotes rules for $n = 0{,}1$ and 2, respectively are shown below in (7)-(9).

$$\int_{x_0}^{x_n} g(x)dx = 2\,h \sum_{i=1}^{n-1} g(x_i) + \frac{h^3}{3} \sum_{i=0}^{n} g''(\eta) \qquad (7)$$

$$\int_{x0}^{x3} g(x)dx = \frac{3h}{2} \left[\sum_{i=1}^{n-2} g(x_i) + \sum_{i=2}^{n-1} g(x_i) \right] + \frac{3h^3}{4} \sum_{i=0}^{n} g''(\eta) \qquad (8)$$

$$\int_{x_0}^{x_n} g(x)dx = \frac{4h}{3}\left[2\sum_{i=1}^{n-3} g(x_i) - \sum_{i=2}^{n-2} g(x_i)\right] + \frac{8h}{3}\sum_{i=3}^{n-1} g(x_i) + \frac{14h^5}{45}\sum_{i=0}^{n} g^{(4)}(\eta) \qquad (9)$$

Where $\eta \in (a,b)$. It is commonly known that the numerical integrators with even n exactly integrate all polynomials of degree at most $n+1$, whereas the same is true for only n degree polynomials for the odd n.

Definition 1.1 Precision: Equation (6) defines a quadrature rule as being of order$p$, for polynomials functions of order at most p, if it achieves exact results i.e., $[Error(f) = 0]$, where p is precision of technique (Chapra, 2012).

Definition 1.2 Contra-Harmonic mean: A number is defined as the Contra-harmonic mean of two numbers, a and b. (Rana, 2022)

$$\xi = \frac{a^2+b^2}{a+b} \qquad (10)$$

3. DERIVATION OF CONTRA-HARMONIC MEAN OPEN NEWTONS COTES RULE BASED ON DERIVATIVES

In this section, a novel family of open Newton-Cotes quadrature procedures is developed for computing definite integrals over (a,b) using Contra-HarmonicMean.

Theorem 1. The derivative-based open mid-point rule using a Contra-Harmonic Mean at n = 0 is:

$$\int_a^b g(x)dx \approx hg\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} g''\left(\frac{a^2+b^2}{a+b}\right) \qquad (11)$$

This method has precision of degree two.

Proof of Theorem 1.

Examine the precise and numerical integrals of the monomials $g(x) = x^2$, x^3 the exact integral:

$$\int_a^b x^2 dx = \frac{b^3-a^3}{3} \qquad (12)$$

$$\int_a^b x^3 dx = \frac{b^4-a^4}{4} \qquad (13)$$

and the mid-point CHMDONC method's numerical integral value when $g(x) = x^2 \; and \; x^3$

$$\int_a^b x^2 dx = \frac{b^3-a^3}{3} \qquad (14)$$

$$\int_a^b x^3 dx = \left[(b-a)\left(\frac{a+b}{2}\right)^3 - 2\left(\frac{a-b}{2}\right)^3\left(\frac{a^2+b^2}{a+b}\right)\right] \qquad (15)$$

Hence, (14) and (15), shows degree of precision of mid-point CHMDONC is two.

Corollary 1: The proposed mid-point CHMDONC rule composite form is as follows:

$$\int_{x_0}^{x_n} g(x)dx = 2\,h\sum_{i=1}^{n-1} g(x_i) + \frac{h^3}{3}\sum_{i=0}^{n} g''\left(\frac{{x_i}^2+{x_{i+2}}^2}{x_i+x_{i+2}}\right) \qquad (16)$$

Theorem 2. *The derivative-based open one-point rule using a Contra-Harmonic Mean at* $n = 1$ *is:*

$$\int_a^b g(x)dx = \frac{3h}{2}\left[g\left(\frac{2a+b}{3}\right) + g\left(\frac{a+2b}{3}\right)\right] + \frac{(b-a)^3}{36} g''\left(\frac{a^2+b^2}{a+b}\right) \qquad (17)$$

This method has precision of degree two.

Proof of Theorem 2.

Examine the precise and numerical integrals of the monomials $g(x) = x^2$, x^3 and confirm that the precision of the mid-point CHMDONC is two. Therefore, when $g(x) = x^2$, get an accurate integral.

$$\int_a^b x^2 dx = \frac{b^3-a^3}{3} \qquad (18)$$

and the mid-point CHMDONC method's numerical integral value when $g(x) = 0$:

$$\int_a^b x^2 dx = \frac{b^3-a^3}{3} \qquad (19)$$

When, $f(x) = x^3$ the exact integral:

$$\int_a^b x^3 dx = \frac{b^4-a^4}{4} \qquad (20)$$

and the one-point CHMDONC method's numerical integral value when $g(x) = x^3$

$$\int_a^b x^3 dx = \left[\frac{\begin{matrix}2(b^5 - a^5) + (a^4 b - ab^4) \\ +5(a^2 b^3 - a^3 b^2)\end{matrix}}{6(a+b)}\right] \qquad (21)$$

Hence, equation (18) and (19), shows degree of precision of one -point CHMDONC is two.

Corollary 2: The proposed one-point CHMDONC rule's composite form is as follows:

$$\int_{x0}^{x3} g(x)dx = \frac{3h}{2}\left[\sum_{i=1}^{n-2} g(x_i) + \sum_{i=2}^{n-1} g(x_i)\right] + \frac{3h^3}{4}\sum_{i=0}^{n} g''\left(\frac{{x_i}^2 + {x_{i+3}}^2}{x_i + x_{i+3}}\right) \qquad (22)$$

Theorem 3. *The derivative-based open two-point rule using a Contra-Harmonic Mean at $n = 2$ is:*

$$\int_a^b g(x)dx = \frac{4h}{3}\left[2g\left(\frac{3a+b}{4}\right) - g\left(\frac{a+b}{2}\right) + 2g\left(\frac{3b+a}{4}\right)\right] + \frac{14h^5}{45} g^{(4)}\left(\frac{a^2+b^2}{a+b}\right) \qquad (23)$$

This method has precision of degree four.

Proof of Theorem 3

Examine the precise and numerical integrals of the monomials $g(x) = x^4$, x^5 and confirm that the precision of the mid-point CHMDONC is two. Therefore, when $g(x) = x^4$, get an accurate integral.

$$\int_a^b x^4 dx = \frac{b^5 - a^5}{5} \qquad (24)$$

and the two-point CHMDONC method's numerical integral value when $g(x) = x^4$:

$$\int_a^b x^4 dx = \frac{b^5 - a^5}{5} \qquad (25)$$

when $g(x) = x^5$ the exact integral:

$$\int_a^b x^5 dx = \frac{b^6 - a^6}{6} \qquad (26)$$

and the two-point CHMDONC method's numerical integral value when $f(x) = x^5$

$$\int_a^b x^5 dx = \left[\frac{\begin{matrix}71(b^7 - a^7) + 147(a^2 b^5 - a^5 b^2) \\ +24(a^4 b^3 - a^3 b^4) + 15(ab^6 - 15a^6 b)\end{matrix}}{384(a+b)}\right] \qquad (27)$$

Hence equation (26) and (27) show that the precision degree is four in the case of the two-point CHMDONC scheme.

Corollary 3: The proposed two-point CHMDONC rule's composite form is as follows:

$$\int_{x_0}^{x_n} g(x)dx = \frac{4h}{3}\left[\begin{matrix} 2\sum_{i=1}^{n-3} g(x_i) \\ -\sum_{i=2}^{n-2} g(x_i) + 2\sum_{i=3}^{n-1} g(x_i)\end{matrix}\right] + \frac{14h^5}{45}\sum_{i=0}^{n} g^{(4)}\left(\frac{{x_i}^2 + {x_{i+4}}^2}{x_i + x_{i+4}}\right) \qquad (28)$$

4. THE ERROR TERMS OF TWO-POINT CONTRA-HARMONIC MEAN-DERIVATIVE-BASED OPEN NEWTON-COTES RULE

Definition 3.1 There are generally both local and global terms in a rule with precision p. [42].

$$local\ error = ch^{(p+2)} g^{(p+1)}(\varepsilon) \qquad (29)$$

$$Global\ error = (b-a)h^{(p+1)} g^{(p+1)}(\eta) \qquad (30)$$

Where, $\varepsilon \in (a,b) and\ \eta \in (a,b)$, c is constant, and h is the step-size.

Theorem 4. *The derivative-based rule using a contra Harmonic Mean at $n = 0$ with local error terms is:*

$$\int_a^b g(x)dx = 2hg\left(\frac{a+b}{2}\right) + \frac{h^3}{3} g''\left(\frac{a^2+b^2}{a+b}\right) - \frac{1}{48}\frac{(b-a)^5}{(b+a)}\ g^{(3)}(\xi) \qquad (31)$$

In which $\xi \in (a,b)$. This method's local accuracy order is four, and its error term is:

$$e[CHMDONC]_{(n=0)} = -\frac{1}{48}\frac{(b-a)^5}{(b+a)}\ g^{(3)}(\xi) \quad (32)$$

Proof of theorem 4

Since $[CHMDONC]_{(n=0)}$has a precision of two, we use the 3rd order derivate term in therefore, we take the third order derivate term in Taylor's series of $g(x)$ about $x = x_0$ is. i.e., $g(x) = \frac{x^3}{3!}$. The difference between the exact and numerical values of the integral at a specific step is known as the local error in numerical integrals. Consequently, with this definition, we have:

$$e[CHMDONC]_{(n=0)} = \left[Exact\left(\frac{x^3}{3!}; a, b\right) - CHMDONC_{(n=0)}\left(\frac{x^3}{3!}; a, b\right)\right] g^{(3)}(\xi) \qquad (33)$$

So that we have,

$$[CHMDONC]_{(n=0)}\left(\frac{x^3}{3!};a,b\right) = \frac{1}{3!}\left[(b-a)\left(\frac{a+b}{2}\right)^3 - 2\left(\frac{a-b}{2}\right)^3\left(\frac{a^2+b^2}{a+b}\right)\right] \qquad (34)$$

Hence from (31) the error term is:

$$e[CHMDONC]_{(n=0)} = -\frac{1}{48}\frac{(b-a)^5}{(b+a)}\, g^{(3)}(\xi) \quad (33)$$

Corollary 4: *CHMDONC of mid-point global error term is:*

$$Global\ error\ [CHMDONC]_{(n=0)} = \frac{2}{3}\frac{(b-a)}{(b+a)} h^4 g^{(3)}(\eta) \qquad (35)$$

In which $\eta \in (a,b)$. Hence the proposed method's global accuracy is three.

Theorem 5. *The derivative-based rule using a Contra-Harmonic Mean at $n=1$ with local error terms is:*

$$\int_a^b g dx = \frac{3h}{2}\left[g\left(\frac{2a+b}{3}\right) + g\left(\frac{2b+a}{3}\right)\right] + \frac{3\square^3}{4} g''\left(\frac{a^2+b^2}{a+b}\right) - \frac{1}{72}\frac{(b-a)^5}{(b+a)}\, g^{(3)}(\xi) \qquad (36)$$

In which $\xi \in (a,b)$. This method's local accuracy order is four, and its error term is:

$$e[CHMDONC]_{(n=1)} = -\frac{1}{72}\frac{(b-a)^5}{(b+a)}\, g^{(3)}(\xi) \quad (37)$$

Proof of Theorem 5.

Since $[CHMDONC]_{(n=1)}$ has a precision of two, we use the 3rd order term's coefficient from the power series development of continuous and successively smooth $f(x)$ with the help of Taylor's expansion around $x = x_0$ is. i.e. $\frac{x^3}{3!}$. The difference between the exact and numerical values of the integral at a specific step is known as the local error in numerical integrals. Consequently, with this definition, we have:

$$e[CHMDONC]_{(n=1)} = \left[Exact\left(\frac{x^3}{3!};a,b\right) - CHMDONC_{(n=1)}\left(\frac{x^3}{3!};a,b\right)\right] g^{(3)}(\xi) \qquad (38)$$

So that we have,

$$[CHMDONC]_{(n=1)}\left(\frac{x^3}{3!};a,b\right) = \frac{1}{3!}\left[\frac{2(b^5-a^5)+(a^4b-ab^4) +5(a^2b^3-a^3b^2)}{6(a+b)}\right] \qquad (39)$$

Hence from (37) the error term is:

$$e[CHMDONC]_{(n=1)} = -\frac{1}{72}\frac{(b-a)^5}{(b+a)}\, g^{(3)}(\xi) \quad (40)$$

Corollary 5: *CHMDONC of one-point global error term is:*

$$Global\ error\ [CHMDONC]_{(n=1)} = \frac{27}{8}\frac{(b-a)}{(b+a)} h^4 g^{(3)}(\eta) \qquad (41)$$

In which $\eta \in (a,b)$. Hence the proposed method's global accuracy is three.

Theorem 6. *The derivative-based rule using a contra Harmonic Mean at $n=2$ with local error terms is:*

$$\int_a^b g(x)dx = \frac{4h}{3}\left[\begin{matrix} 2g\left(\frac{3a+b}{4}\right) \\ -g\left(\frac{a+b}{2}\right) + 2g\left(\frac{3b+a}{4}\right)\end{matrix}\right] + \frac{14h^5}{45} g^{(4)}\left(\frac{a^2+b^2}{a+b}\right) + \frac{7}{276480}\frac{(b-a)^7}{(b+a)} g^{(5)}(\xi) \ (42)$$

In which $\xi \in (a,b)$. This method's local accuracy order is six, and its error term is:

$$e[CHMDONC]_{(n=2)} = \frac{7}{276480}\frac{(b-a)^7}{(b+a)} g^{(5)}(\xi) \ (43)$$

Proof of Theorem 6.

Since $[CHMDONC]_{(n=2)}$ has a precision of four, we use the 5th order term's coefficient from the power series development of a continuous and successively smooth $f(x)$ with the help of Taylor's expansion around $x = x_0$, i.e. $\frac{x^5}{5!}$. The difference between the exact and numerical values of the integral at a specific step is known as the local error in numerical integrals. Consequently, with this definition, we have:

$$e_{CHMDONC(n=2)} = \left[Exact\left(\frac{x^5}{5!};a,b\right) - CHMDONC_{(n=2)}\left(\frac{x^5}{5!};a,b\right)\right] g^{(5)}(\xi) \qquad (44)$$

So that we have,

$$CHMDONC_{(n=2)}\left(\frac{x^5}{5!};a\,,b\right)$$

$$=\frac{1}{5!}\left[\frac{\begin{matrix}71(b^7-a^7)+147(a^2b^5-a^5b^2)\\+24(a^4b^3-a^3b^4)+15(ab^6-15a^6b)\end{matrix}}{384(a+b)}\right] \quad (45)$$

Hence from (42) the error term is:

$$e[CHMDONC]_{(n=2)}=\frac{7}{276480}\frac{(b-a)^7}{(b+a)}g^{(5)}(\xi) \quad (45)$$

Corollary 6: *CHMDONC of two-point global error term is:*

$$Global\ error\ [CHMDONC]_{(n=2)}$$
$$=\frac{56}{135}\frac{(b-a)}{(b+a)}h^6g^{(5)}(\eta) \qquad (46)$$

In which $\eta \in (a, b)$. Hence the proposed method's global accuracy is five.

5. NUMERICAL RESULTS AND DISCUSSIONS

In this section, some numerical tests have been obtained on developed quadrature rules, which were verify validity of theoretical results. For every scheme, three numerical problems from literature are solved, using MATLAB (R2014b) Software. Analysis of methods is divided into three sections. In first section one the theoretical error analysis of the methods is assessed, in which the local and global error terms, the precision and accuracy are derived. A quadrature rule may offer acceptable accuracy in fewer steps because of the increased number of functional evaluations at each integration step, but it may also be more computationally costly and less more efficient than alternative methods. In regard to this, the next section is devoted to the estimation of total computing cost at each integration step that reaches the predetermined error tolerance. In the last section, the observed/numerical accuracy order by the help of (47) with reference to the study of Burg, C.O (2012).

$$COCp=\frac{\log\left|\frac{N(2h)-N(0)}{N(h)-N(0)}\right|}{\log(2)} \qquad (47)$$

Whereas $N(0)$ denotes the exact definite integral, and $N(h)$ and $N(2h)$ are the approximate results using the numerical integrals with intevval-width of h and $2h$, respectively.

Finally, the absolute errors between the new modified techniques and the ones that already exist are compared using line plots. The effectiveness of the suggested derivative-based methods against classical methods is shown for the following integrals from the literature.

$$I_1=\int_0^1 x^5cosxdx=-1.348691911046506$$

$$I_2=\int_0^{\pi/4} e^{cosx}dx=1.939734850623649$$

$$I_3=\int_0^1 xln(1+x)/(1+x\text{^}2)dx=0.162865005917789$$

The Tables 1-3 illustrates the observed/numerical accuracy orders in the cases of the new developed schemes, which validates the theoretical outcomes derived in theorems, previously. The number of strips m are shown in the first and second columns of each 2-3, respectively. The computing order of accuracy of the regular ONC quadrature rules is indicated by the columns below each ONC heading, whereas the computational order of accuracy of the suggested Contra-Harmonic Mean derivative-based rules is indicated by the columns below the CHMDONC heading. Hence the order of accuracy of proposed mid-point, one –point and two-point method so obtained is 4, 4 and 6 respectively. Therefore, the results show that the updated approaches have a two-order boost in accuracy.

Table1. COCp of all ONC mid point rules

m	ONC I_1	CHONC I_1	ONC I_2	CHONC I_2	ONC I_3	CHONC I_3
1	NA	NA	NA	NA	NA	NA
2	1.3061	3.3696	2.0537	3.9001	1.7633	1.8802
4	2.0806	4.7470	2.0129	4.1025	1.8611	3.3766
8	2.0313	4.1258	2.0032	4.0960	1.9718	3.7563
16	2.0084	4.0096	2.0007	4.0601	1.9931	3.8353
32	2.0021	3.9991	2.0001	4.0332	1.9983	3.8517

Table 2. COCp of all ONC one-point rules

m	ONC I_1	CHONC I_1	ONC I_2	CHONC I_2	ONC I_3	CHONC I_3
1	NA	NA	NA	NA	NA	NA
2	1.5367	3.2144	2.0443	3.8981	0.2180	3.1389
4	3.6068	4.6913	2.0106	4.1033	1.8884	3.6651
8	2.0260	4.1109	2.0026	4.0969	1.9768	3.7898
16	2.0070	4.0073	2.0006	4.0607	1.9943	3.8263
32	2.0017	3.9988	2.0001	4.0335	1.9986	3.8438

Table 3. COCp of all ONC two-point rules

m	ONC I_1	CHONC I_1	ONC I_2	CHONC I_2	ONC I_3	CHONC I_3
1	NA	NA	NA	NA	NA	NA
2	4.8122	8.3690	4.0893	5.6645	5.1968	5.0772
4	3.4197	6.7058	4.0228	6.0588	4.2202	4.4409
8	3.9065	5.4172	4.0056	6.0936	4.0300	5.6691
16	3.9787	5.7107	4.0014	6.0639	4.0070	5.8913
32	3.9948	5.8347	4.0003	6.0358	4.0017	5.9162

Table 4 presents the overall computational cost (i.e., the total number of functions and their derivatives evaluations involved in the quadrature rule). Table 5 displays the overall computational time. The findings of the analysis indicated that the modified CHMDONC methods are less expensive than the regular approaches. The findings of the analysis indicated that the modified CHMDONC methods are less expensive than the regular approaches.

Table 4. Comparison of numerical cost/burden

Methods	I_1	I_2	I_3
Mid-point ONC	1412499	19200	10206
Mid-point CHMDONC	2466	252	452
One-point ONC	3477306	47025	16666
One-point CHMDONC	3393	342	609
Two-point ONC	2136	135	168
Two-point CHMDONC	324	56	72

Table 5. Comparison of computational time

Methods	I_1	I_2	I_3
Mid-point ONC	2.1138	0.5969	0.5031
Mid-point CHMDONC	1.3388	0.4210	0.4096
One-point ONC	1.9198	0.4326	0.4543
One-point CHMDONC	1.3282	0.4250	0.4544
Two-point ONC	1.3730	0.4792	0.5968
Two-point CHMDONC	1.2443	0.4421	0.5609

The figures 1-3 represents the absolute errors that are computed to analyze and contrast the outcomes of the suggested CHMDONC and traditional ONC approaches vs the number of strips (i.e., from 1 to 20) for the three integrals $I_1, I_2, \& I_3$ respectively. The results produced by the new techniques verify that the new methods possess lesser errors than the classical techniques.

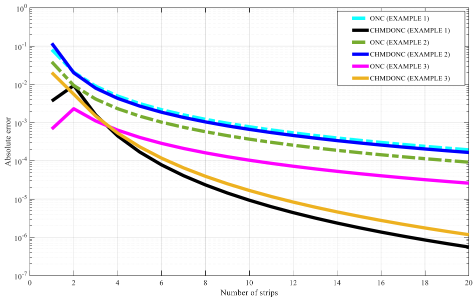

Figure 1. Error graphs for all Mid-point ONC

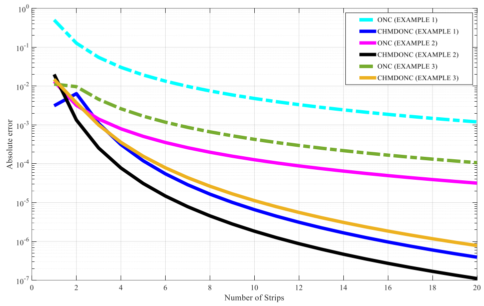

Figure 2. Error graphs for all One-point ONC

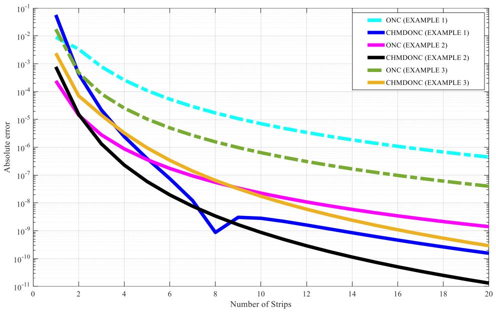

Figure 3. Error graphs for all Two-point ONC

5. CONCLUSION

This study presents a novel approach to developing a family of open quadrature rules based on derivatives utilizing the Contra-Harmonic Mean. The proposed methods demonstrate a higher order of accuracy and enhanced precision, thereby outperforming existing techniques. The derivation of error terms for all modified methods provides a comprehensive understanding of their convergence properties. Numerical experiments validate the efficacy of the updated CHMDONC rules, showcasing significant reductions in computational cost and absolute error. These findings underscore the potential of the proposed quadrature schemes to refine numerical integration methods. Furthermore, the flexibility of these schemes to accommodate higher intervals and diverse applications, including derivatives of statistical means, positions them as valuable tools for future research and practical implementations.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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