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DEVELOPMENT OF NEW METHODS OF PROVING RECIPROCITY THEOREM AND DERIVING CAUCHY — REIMANN EQUATIONS IN POLAR FORM

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Abstract

New and easy methods have been developed to prove Reciprocity theorem and to derive Cauchy–Riemann equations in Polar form.

Introduction

Making postulates, discovery of laws or presentation of new theory is some thing for which one should be grateful to the Scientists concerned. For example Newton's laws of motion, Ohm's law etc. are so important that we can not move a step ahead without them in this Scientific world.

Nevertheless, the development of new methods of proving laws or deriving equations is also significant and carries weight. They are particularly useful when they are easy to understand.

Mathematical Formulations

a. Reciprocity theorem

The theorem states that in any bilateral network, the ratio of the applied voltage in the first loop denoted by "A" to the current in the second loop denoted by "B" is equal to the ratio of the applied voltage in the second loop (loop B) to the current in the first loop (loop A).

If E_1 is the applied voltage in the loop "A" and I_2 is the corresponding current in the loop "B" and E_2 is the applied voltage in the loop "B" and I_2 is the current in the loop "A" then

$$\frac{E_1}{I_2} = \frac{E_2}{I_1}$$

Here we have applied simple loop analysis to prove the theorem. The method is very much clear from the Figs (i) and (ii).

$$r_1 I_2 + r_2 I_2 + r(I_2 - I_4) = 0 \longrightarrow I_2(r_1 + r_2 + r) = r I_4 \longrightarrow \ldots\ldots (1)$$

$$r_3 I_4 + r_4 I_4 (I_4 - I_4) = - E_1$$

or

$$I_4 (r_3 + r_4 + r) = r I_2 - E_1 \ldots\ldots (2)$$

$$r_1 I_3 + r_2 I_3 + r(I_3 - I_1) = - E_2$$

or $I_3(r_1 + r_2 + r) = r I_1 - E_2 \ldots\ldots (3)$

& $r_3 I_1 + r_4 I_1 + r(I_1 - I_3) = 0 \longrightarrow I_1(r_3 + r_4) = r I_3 \longrightarrow \ldots\ldots (4)$

or $I_3 = \dfrac{r_3 + r_4 + r}{r} I_1$ (from equation 4)

Thus equation (3) becomes

$$\frac{(r_3 + r_4 + r)}{r}(r_1 + r_2 + r) I_1 = r I_2 - E_2 \longrightarrow \ldots\ldots (5)$$

From equation (1)

$$I_4 = \frac{r_1 + r_2 + r}{r} I_2$$

Thus equation (3) becomes

$$I_4 = \frac{r_1 + r_2 + r}{4} I_2 \qquad \text{.........(6)}$$

Deviding equation (5) by equation (6), we get:

$$\frac{(r_3 + r_4 + r)}{r} (r_1 + r_2 + r) \times \frac{r I_1}{(r_1 + r_2 + r)(r_3 + r_4 + r) I_2} = \frac{r I_1 - E_2}{r I_2 - E_1}$$

or $$\frac{I_1}{I_2} = \frac{r I_2 - E_2}{r I_2 - E_1}$$

or $$I_1 r I_2 - E_1 I_1 = I_2 r I_1 - E_2 I_2$$

$$E_1 I_1 = E_2 I_2$$

or $$\frac{E_1}{I_2} = \frac{E_2}{I_1}$$ ————————————> Reciprocity theorem

(b) New method of deriving Cauchy-Riemann equations in polar form Consider the compledx function f (Z) represented as

$$f(z) = u(r, \theta) + i V(r, \theta) \qquad \text{.........(1)}$$

where $z = r \cos\theta + i r \sin\theta$ ....(2)

A complex number z may be regarded as vector whose magnitude in cartesian coordinates (x,y) is given as

$$|Z| = \sqrt{x^2 + y^2}$$

This is the method used for calculating the magnitude of a vector. Thus the complex number z expessed in polar coordinates (r , θ) also behaves as a vector because the property of a physical quantity does not change with the change of coordinate system. Hence a change Δz in z also behaves as vector. As very much clear from the figure its components are Δr and $r \Delta \theta$ in the directions of r and θ increasing. Here the direction of r is taken along the real axis and that of θ which is perpendicular to the direction of r is taken as the imaginary axis.

In order to get polar form of Cauchy — Riemann equations we then take

$\Delta Z = \Delta r + i\, r \Delta \theta$ ....(3)

Equation (3) gives an easier method of finding Δz instead of going through un-necessary formal details involving equation (2).

Differential coefficient of f (z) with respect to z is independent of the way on which Δz is made to approach zero. There are two obvious ways in which Δz can be made to approach zero.

They are:

(i) We first let $\Delta \theta = 0$

From equation (3) we have

$\Delta z = \Delta r$

So that $\Delta z \longrightarrow 0$ as $\Delta \theta \longrightarrow 0$

(ii) Next we let $\Delta r = 0$

$\Delta z = i\, r \Delta \theta$ ....(5)

$$f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

From equations (1) & (4)

$$f(z) = \lim_{\Delta r \to 0} \frac{U(r+\Delta r, \theta) + i\, V(r+\Delta r, \theta) - U\{(r,\theta) + i\, V(r,\theta)\}}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{U(r+\Delta r,\theta) - U(r,\theta)}{\Delta r} + \lim_{\Delta r \to 0} \frac{i\{(V(r+\Delta r,\theta) - V(r,\theta)\}}{\Delta r}$$

$$f(z) = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \quad \text{..........(6)}$$

Again from equation (1) & (5)

$$f(z) = \lim_{\Delta r \to \theta} \frac{U(r,\theta+\Delta\theta) + i\, V(r,\theta+\Delta\theta) - \{U(r,\theta + i\, V(r,\theta)\}}{i\, r \Delta \theta}$$

$$= \frac{1}{i} \lim_{\Delta\theta \to 0} \frac{U(r,\theta+\Delta\theta) - U(r,\theta)}{\Delta\theta} + \frac{i}{i} \lim_{r\Delta\theta \to 0}$$

$$\frac{\{V(r,\theta+\Delta\theta)-V(r,\theta)\}}{\Delta\theta}$$

$$= -\frac{i}{r}\frac{\partial U}{\partial\theta} + \frac{i}{r}\frac{\partial V}{\partial\theta} \quad \text{.......... (7)}$$

From equations (6) and (7) after equating real and imaginary parts separtely we get

(i) $\frac{\partial U}{\partial r} = \frac{i}{r}\frac{\partial V}{\partial\theta}$

(ii) $\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial\theta}$ .... (8)

These are the required Cauchy — Riemann equations is Polar form. This method is easy for students to understand.

Conclusion

Reciprocity theorem has been proved and Cauchy –Riemann equations have been derived in the original form by the new methods:

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