Short Communication

ON RIGHT PRIMITIVE DISTRIBUTIVELY GENERATED NEAR-RINGS

Muhammad Shafi
226-B, New Chauburji Park, Lahore, Pakistan

Abstract

A right primitive d.g. near-ring R forms a topological d.g. near-ring Rp having ηp as a complete system of neighbourhoods of 0.

Introduction

It is well known that a primitive d.g. near-ring is a prime d.g. near-ring [Cf. (7)]. In this note we shall show that a right primitive d.g. near-ring forms a topological d.g. near-ring R_p having η_p as a complete system of neighbourhoods of 0. We shall use some standard concepts without explicit definitions.

Preliminaries

It is assumed that the reader is familiar with the basic definitions and elementary properties of near-rings presented in [(1) to (7)]. In (1) A. Frohlich defined an abstract distributively generated (d.g.) near-ring R to be an abstract near-ring which has a subset S of distributive elements generating (R,+). A topological near-ring is an abstract near- ring which is also a Hausdorff toplogical space such that the near-ring is said to be distributively generated if it has a subset S of distributive elements which generated (R,+) topologically.

Throughout this note we shall assume (i) the term near-ring without further qualifications refers to a right near-ring with identity, (ii) W denotes an additive group not necessarily abelian, and (iii) if R is a near-ring then T(R) denotes the set of all distributive elements in R and $T_0(R)$ denotes the set of all non-zero distributive elements in R. Now let R be a topological d.g. near-ring with identity 1, generated by the distributive semigroup S.

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Definition 1. A topological right R-group is a topological group W together with a continuous map $(w,x) \longrightarrow wx$ of $W \times R \longrightarrow W$ such that

(i) $(w_1+w_2)\ x=w_1\ x+w_2x$ for all x of R and w_1, w_2 of W

(ii) $w(xy) = (wx)y$ for all x,y of R and w of W

(iii) $w.1 = w$ for all w of W.

Definition 2. An element d of the topological right R-group W is said to be distributive if $d(x+y) = dx + dy$ for all x,y of R.

We will denote the set of all distributive elements in W by V_0.

Definition 3. A topological d.g. right (R,S)-group is a group W such that

(i) W is a topological right R-group,

(ii) there exists a subset V of V_0 such that

$VS \subseteq V \cup \{0\}$ and V generates W (as a group) topologically. If we wish to specify the sets S and V we shall speak of the topological d.g. right (R,S)-group (W,V); otherwise we shall use the term topological d.g. right R-group W.

Definition 4. A subgroup H of a topological d.g. right R-group W is said to be

(i) a right R-subgroup if $HR \subseteq H$,

(ii) a normal right R-subgroup if H is a normal subgroup and $HR \subseteq H$

(iii) a d.g. right R-subgroup if $HR \subseteq H$ and $V_0 \cap H$ generates H (as a group) topologically.

A d.g. R-subgroup of the d.g. right R-group (R,+) will be called a d.g. right R-module.

A right ideal of R which is also a d.g. right R-module will be called a d.g right ideal.

Definition 5. A continuous group homomorphism ϕ of a topological right R-group W into a topological right R-group W′ is called a continuous right R-homomorphism if $\phi\ (wx) = \phi\ (w)x$ for all x of R and w of W.

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Definition 6. A d.g. R-group W is said to be irreducible if (i) it has no non-trivial closed normal right R-subgroups and (ii) it has no non-trivial d.g. right R-subgroups.

Definition 7. A d.g. right R-group is said to be faithful if $\{x \in R;\ Wx = 0\} = 0$.

Definition 8. A topological d.g. near-ring R is said to be right primitive if there exists a faithful, irreducible d.g. right R- group.

For the following we shall assume that (i) R is a right primitive d.g. near-ring (ii) W is a faithful, irreducible d.g. right R- group having $\sum$ as a complete system of neighbourhoods of 0 and V_0 neighbourhoods the set of all distributive elements and (iii) K is a subset of V_0 and $r(K) = \{x \in R: Kx = 0\}$ with $r(\{v\})$ being written as $r(\underline{v})$.

Definition 9. The centraliser $S^{(1)}$ of W is the set of all right R-endomorphisms of W.

Proposition 1.

For all non-zero elements v of V_0, r(v) is a closed right ideal. Also r(v) is a maximal d.g. right R-module and a maximal closed right ideal of R.

Proof. Now vR is a non-zero d.g. right R-subgroup of W and W is irreducible. Consequently vR=W. The mapping $x \longrightarrow v_x$ of $(R, +) \longrightarrow W$ is clearly a right R-homomorphism with kernel r(v). Thus r(v) is a closed right ideal of R. Since W is irreducible, r(v) is a maximal d.g. right R-module and a maximal closed right ideal.

From the proof of the above proposition it follows that for a given v_0 of V_0 there exists a continuous (1-1) right R- homomorphism from $R\text{-}r(v_0)$ onto W and since by proposition 1.4, Corollary of (9), $R\text{-}r(v_0)$ is a d.g. right R-group, we may (and do) introduce a topology into W such that W is a faithful, irreducible d.g. right R-group which is right R-isomorphic to $R\text{-}r(v_0)$ where v_0 is a fixed non-zero element of V_0. We also note from the above proposition that since the map $\bar{z} \longrightarrow v\bar{z}$ is a right R- isomorphism of $R\text{-}r(v_0)$ onto W, $\bar{z}$ is distributive in $R\text{-}r(v_0)$ if and only if $v\bar{z}$ is distributive in W.

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Now the proof of this proposition is very similar to the proof of th proposition 3.5 in (8) and will be omitted.

Proposition 2.

$S^{(1)}\setminus\{0\}$ is a group of regular automorphisms of W.

From now on we shall, for the sake of convenience, restrict ourselves to the case when R is a right primitive d.g near-ring having a discrete W. In particular, R may be discrete or have a neighbourhood system consisting of right ideals.

Let Γ_p be the set of all non-empty finite subsets of V_0 and η_p the set of all $N_k=\{x\in R: Kx=0\}$, $K\in\Gamma_p$. Clearly N_k is a right ideal for each $K\in\Gamma_p$.

Proposition 3.

N_k is an open set in R.

Proof. Let υ be a variety of abstract (R_1,S_1)-groups (Cf.(5)).

For each $v\in K$, there exists $D_v\in\upsilon$ such that $vD_v=0$. Let $D=\bigcap_{v\in K} D_v$. Then $KD=0$,i.e. $D\subseteq N_k$. Thus N_k is an open set in R.

Proposition 4.

R forms a topological d.g. near-ring R_p having η_p as a complete system of neighbourhoods of 0.

Proof. (i) $\cap N_k=0$.

(ii) Given $N_k\in\eta_p$, $N_k+(-N_k)\subseteq N_k$

(iii) Given $N_k\in\eta_p$ and $x\in N_k$, $N_k+x\subseteq N_k$.

(iv) Given $N_k\in\eta_p$ and $x\in R_p$, $-a+N_k+a\subseteq N_k$

(v) Given N_{k_1} and $N_{k_2}\in\eta_p$ Let $K=K_1\cup K_2$.

Then $N_k \in h_p$ and $N_k \subseteq N_{k_1} \cap N_{k_2}$.

(vi) Let $s \in S$ and $x,y \in R_p$ and $N_k \in \eta_p$ be given

Choose $K_1 = K \cup K_s$ (Note $K_s \subseteq V_0$)

Then for $v \in K$ we have

$$v\,[-sy+(s+N_{k_1})\,(y+N_{k_1})] \subseteq -vsy+vs(y+N_{k_1}) = -vsy+vsy = 0$$

$$(s+N_{k_1})(y+N_{k_1}) \subseteq sy+N_k.$$

Suppose $x_1, x_2 \in R_p$ such that given $y \in R_p$ and

$N_k \in \eta_p$ there exists $N_{k_i} \in \eta_p$ with

$N_{k_i} \subseteq N_k$ and $(x_i+N_{k_i})(y+N_{k_i}) \subseteq x_iy+N_k$, $i=1,2$

Now if $K_3 = K_1 \cup K_2$ then

$$(x_1+x_2+N_{k_3})(y+N_{k_3}) = (x_1+N_{k_3})(y+N_{k_3})+(x_2+N_{k_3})(y+N_{k_3})$$

$$\subseteq (x_1+N_{k_1})(y+N_{k_1})+(x_2+N_{k_2})(y+N_{k_2})$$

$$\subseteq (x_1y+N_k+x_2y+N_k = (x_1+x_2)y+N_k.$$

Now since $(x_1+N_{k_1})(Y+N_{k_1}) \subseteq x_1y+N_k$ we have on multiplying both sides on the left by $(-e)$

$$-(x_1+N_{k_1})(y+N_{k_1}) \subseteq -N_k \; -x_1y = (-x_1)y+N_k$$

$$(-x_1+N_{k_1})(y+N_{k_1}) \subseteq (-x_1)y+N_k$$

Thus if R' is the subgroup of R_p generated by S then map $(x,y) \longrightarrow xy$ of $R' \times R_p \longrightarrow R_p$ is continuous.

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By proposition 1.3 of (8), R' is densely embedded in R_p

Now let $x,y \in R_p$ and $N_k \in \eta_p$

be given. Then there exists $x' \in R$ such that $x-x' \in N_k$.

Choose $N_{k_1} \in \eta_p$ such that $x'(y+N_{k_1}) \subseteq x'y + N_k$.

Since $N_k (y+N_{k_1}) \subseteq N_k$ we have $(x-x')(y+N_{k_1}) \subseteq N_k$

i.e., $x(y+N_{k_1}) \subseteq N_k + x'(y+N_{k_1}) \subseteq x'y + N_k$

$= x'y - xy + xy + N_k$

$\subseteq N_k + xy + N_k = xy + N_k$.

The map $(x,y) \longrightarrow xy$ of $R_p \times R_p \longrightarrow R_p$ is continuous and hence R_p is a d.g. near-ring.

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