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# Analytical Solution of Lift for Thin Film Flow for Phan Thien Tanner Fluid

S. CHANNER*, K. N. MEMON**[++], A. A. GHOTO**, A. M. SIDDIQUI***, S. F.SHAH*

*Department of BSRS, MUET, Jamshoro, Sindh, Pakistan

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**Abstract:** The present work analyses the study of thin film flow of a steady, incompressible, non-isothermal under the influence of variable viscosity for Phan Thien Tanner fluid on a vertical belt. We have derived the basic governing non-linear differential equation as of the continuity and momentum equation. Then we have used Perturbation technique to solve resulting equation. Reynold model is used for temperature dependent viscosity. The upper convected Maxwell (UCM), linear PTT (LPTT) and quadratic PTT (QPTT) models have been solved from this considered model. Interpretation for "velocity profile, temperature distribution, volume flow rate and average velocity" has been obtained in this case. Consequence of distinct parameters on "velocity profile" and "temperature distribution" are shown graphically and therefore the comparison is also given for velocity and temperature distribution for all the special cases of PTT by using tables.

**Keywords**: Thin Film flow, Phan Thien Tanner, Reynold model, Perturbation technique, Heat transfer.

## 1. INTRODUCTION

In recent years, the most attention has been gained by non-Newtonian fluids in the several biological and industrial technological: mostly in chemical industries, bioengineering and material processing. Here insufficient belongings of non-Newtonian fluids, for example, drilling mud, toothpaste, greases, blood, paints, clay coatings, polymer melts etc. It is an extensive class of fluids so; no single model can deal with each property of such fluids as is done by Newtonian fluids (described by the well-known Navier-Stokes equation). Regarding to this several fundamental equations have been considered to anticipate the physical structure and nature of such fluids for various materials (Abel *et al.*, 2014; Deshpande, and Barigou, 2001; Memon, *et al.*, 2014; Memon, *et al.*, 2018). It is so, difficult regards study in against that of a Newtonian fluid, because of that is a nonlinear connection between the rate of restraining and shear stress. The Phan Thien Tanner model has been designed to a large extent as the class of non-Newtonian fluid, by the reason of mathematical ease and general industrial applications (Memon, *et al.*, 2014; Yong-Li Chen, *et al.*, 2009; Schowalter, 1978).

Here our principal concentration is investigation of thin layer flow concerning a PTT fluid with the temperature dependent fluid viscosity by the use of Reynold model (Phan-Thien, Tanner, 1977). In a thin film flow, the liquid is partly restricted through one boundary whereas the other boundary can relate with other liquid, e.g., air. Formation of thin films is based on three fundamental expressions namely, centrifugal forces, gravitational forces and surface tension. The study of thin layer flow is significant concerning chemical processing. Examples of everyday life are the flow of a tear films in the eye membrane, paint down a wall and rainwater running down along a window (Siddiqui, *et al.*, 2006, 2012, 2013, 2016, Bird. 1987). Here, in our work, fluid is considered viscoelastic with variable viscosity consistent to Phan Thein Tanner fluid (Sasui*et al.*, 2018; Mohyuddin *et al.*,2005); (Mercant and Atalık 2012). We have observed theoretically the flow of thin film for a Phan Thein Tanner fluid model concerning lift problem on aupright belt. Three estates are examined, namely QPTT, LPTT and UCM. As the best of our insight the results by using perturbation methodis not accounted anywhere.

The plan of the research article is ordered as follows: Section 2 holds the basic governing equations of Phan Thein Tanner model and section number 3 covers problem considerable and solution. Results and discussion be specified intothe section 4 and in Section number 5concluding remarks are given.

## 2 GOVERNING EQUATIONS

Essential governing equations for incompressible Phan Thien Tanner Fluid , includin thermal effects are:

$$\nabla \cdot \mathbf{V} = 0, \quad (1)$$

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{b} - \nabla p + \nabla \cdot \mathbf{T}, \quad (2)$$

[++]Corresponding author Email: knmemon@quest.edu.pk, kamrannazir025@yahoo.com, kamrannazirmemon@gmail.com

** Department of Mathematics and Statistics, Quaid –e-Awam University of Engineering Science and Technology, Nawabshah, Pakistan

***Pennsylvania State University, Campus York, Edgecombe 17403, United States of America

$$\rho C_p \frac{D\theta}{Dt} = k\nabla^2\theta + \frac{1}{2}tr(\mathbf{TA}_1) \quad (3)$$

The symbol $p$ be the dynamic pressure, $\rho$ stands fordensity, $\mathbf{V}$ represent velocity field, $\mathbf{T}$ be extra stress tensor, $\mathbf{b}$ represent to body force, $\theta$ for temperature distribution, $\eta$ bethe viscosity coefficient, $k$ represent to thermal conductivity,operator $\frac{D}{Dt}$ denotes the material derivative, $C_p$ be the specific heat constantand $\mathbf{A_1}$ be the 1st Rivlin Ericksen tensor, which is represented as

$$\mathbf{A}_1 = (\nabla\mathbf{V})^T + \nabla\mathbf{V}. \quad (4)$$

For PTT fluid model constitutive equations (Faraz, N., Lei, H., Khan, Y, 2015; A. M. Siddiqui, et al., 2006;Sasui*et al.*, 2018) is given as

$$f(tr\mathbf{T})\mathbf{T} + \lambda\overset{\nabla}{\mathbf{T}} = \eta\mathbf{A}_1, \quad (5)$$

Here $\lambda$ is the relaxation time and symbol for upper convected derivativeis $\overset{\nabla}{\mathbf{T}}$, which is characterized as:

$$\overset{\nabla}{\mathbf{T}}. = -\left((\nabla\mathbf{V})^t\mathbf{T} + \mathbf{T}(\nabla\mathbf{V})\right) + \frac{D\mathbf{T}}{Dt} \quad (6)$$

For PTT fluid model, there are three special cases which are commonly used as

1. Upper Convected Maxwell (UCM) Model

$$f(tr\mathbf{T}) = 1 \quad (7)$$

2. Linear PTT (LPTT) Model

$$f(tr\mathbf{T}) = 1 + \frac{\varepsilon\lambda}{\eta}tr\mathbf{T} \quad (8)$$

3. Quadratic PTT Model (QPTT Model)

$$f(tr\mathbf{T}) = 1 + \frac{\varepsilon\lambda}{\eta}trT + \frac{\delta_1}{2}\left(\frac{\varepsilon\lambda}{\eta}trT\right)^2 \quad (9)$$

Where $\varepsilon$ is parameter represent the "elongational behavior" of the fluid model. Phan Thien Tanner fluid flow model be there shear thinning and exponential PTT fluid model is further thinner than the linear PTT fluid model. Shear thinning effects are directly related to the value of $\varepsilon$. Elongational viscosity is inversely proportional to $\varepsilon$ (Siddiqui, *et al.*, 2016).

## 3 CONSIDEARABLE PROBLEM AND ITS SOLUTION

Let we take a vessel full of an incompressible Phan Thien Tanner fluid with variable temperature dependent viscosity. An extensive belt moves upward with constant velocity $U$ and gets a layer of Phan Thien Tanner fluid of a uniform thickness $\delta$ during motion but due to gravity fluid tries drain down to the belt. Consider the flow of a fluid is parallel, laminar and steady. We have assumed $p$ as gauge pressure.

Here, we have considered *xy*-coordinate system with the end goal that "*y*-axis" is alongside the belt into the upward direction and "*x*-axis" is normal on belt. In like manner, we expect that

$$\mathbf{V} = [0, v(x), 0], \quad \mathbf{T} = \mathbf{T}(x), \quad \theta = \theta(x) \text{ only}$$

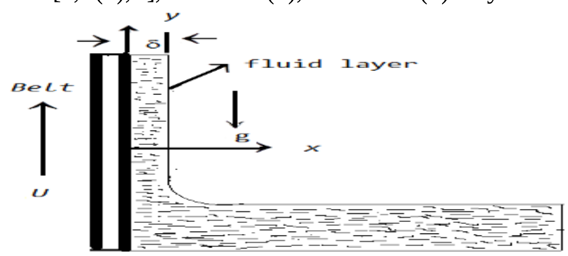


**Fig. 1. Physically geometry for fluid flow through vertical belt, which is moving through container.**

Related boundary conditions concerning the proposed problem are

$$T_{xy} = 0 \quad and \quad \frac{d\theta}{dx} = 0 \quad at \quad x = \delta, \quad (11)$$

$$v = U \quad and \quad \theta = \theta_0 \quad at \quad x = 0 \quad (12)$$

By using equation (10) into continuity equation (1) remains identically fulfilled and non-zero equation of motion at atmospheric pressure and energy equation after using the value of 1st Rivlin Ericksen tensor is

$$0 = -\rho g + \frac{dT_{xy}}{dx} \quad (13)$$

$$0 = k\frac{d^2\theta}{dx^2} + T_{xy}\frac{dv}{dx} \quad (14)$$

Intigrate to equation (13) W.R.T., $X$ and after applying free space boundary condition, we get

$$T_{yx} = -\rho g(\delta - x) \quad (15)$$

Inserting equation (10) into the equations (4-6), after considerable calculations once we obtain:

$$f(tr\mathbf{T})T_{xx} = f(tr\mathbf{T})T_{zz} = f(tr\mathbf{T})T_{zx} = 0 \quad (16)$$

$$f(tr\mathbf{T})T_{yz} = T_{zx}\frac{dv}{dx} \quad (17)$$

$$f(tr\mathbf{T})T_{xy} = \eta\frac{dv}{dx} + \lambda T_{xx}\frac{dv}{dx} \quad (18)$$

$$f(tr\mathbf{T})T_{yy} - 2\lambda T_{yx}\frac{dv}{dx} = 0 \quad (19)$$

Since $f(tr\mathbf{T})$ has one of the values given in equation (7) – (9), therefore $f(tr\mathbf{T}) \neq 0$. Which implies that

$$T_{xx} = T_{zz} = T_{xz} = 0 \quad (20)$$

By applying these values from equation (20) into the equations (17) - (19) we get

$$T_{yz} = 0 \quad (21)$$

$$f(tr\mathbf{T})T_{xy} = \eta \frac{dv}{dx} \quad (22)$$

$$f(tr\mathbf{T})T_{yy} = 2\lambda T_{yx} \frac{dv}{dx} \quad (23)$$

Joining equations (22) and (23), we get

$$T_{yy} = \frac{2\lambda T_{xy}^2}{\eta}. \quad (24)$$

Which is relationship among ordinary stresses and shear and that is Parabola with the axis at $T_{xy} = 0$ and a vertexof parabola at the origin. The focus of parabola is $\left(0, \frac{1}{8}\left(\frac{\eta}{\lambda}\right)\right)$ which lies on axis of $T_{yy}$ and opening is at upward. $T_{yy} = 0$ is tangent to its vertex and as well asparabola. After using equation (20) and (24), the tace of trace of extra stress tensor can be defined as:

$$tr\mathbf{T} = \frac{2\lambda T_{xy}^2}{\eta} \quad (25)$$

Using shear stress (15) in equation (24), normal stress is given by

$$T_{yy} = \frac{2\lambda\rho^2 g^2 (\delta - x)^2}{\eta} \quad (26)$$

Rearranging equation (22) after substitute the value of $T_{yx}$ from equation (15), we can write it as

$$\frac{dv}{dx} = -\frac{\rho g\ f(tr\,\mathbf{T})(\delta - x)}{\eta} \quad (27)$$

Substituting equation number (15) in (14), we obtain

$$0 = k\frac{d^2\theta}{dx^2} - \rho g(\delta - x)\frac{dv}{dx} \quad (28)$$

Analytical solutions of this equation along with the boundary conditions (11) -(12) are obtained for three cases (7)-(9) in the following subsections:

### 3.1 Solutions for UpperConvected Maxwell (UCM) Model:

By using UCM model (7) in (27), we get;

$$\frac{dv}{dx} = -\frac{\rho g\,(\delta - x)}{\eta} \quad (29)$$

making dimensionless by significant the following dimensionless parameters:

$$\frac{\eta}{\eta_0} = \eta^*, \quad \frac{v}{U} = v^*, \quad \frac{x}{\delta} = x^*, \quad \theta^* = \frac{\theta - \theta_0}{\theta_1 - \theta_0} \quad (30)$$

Hence equation number (28) and (29), after dropping '*' becomes

$$\eta\frac{dv}{dx} = -S_t(1-x) \quad (31)$$

$$0 = \frac{d^2\theta}{dx^2} - S_t B_r (1-x)\frac{dv}{dx} \quad (32)$$

Where $S_t = \frac{\rho g \delta^2}{\eta_0 U}$ denotes Stokes number and $B_r = \frac{\eta_0 U^2}{k(\theta_1 - \theta_0)}$ is Brinkman number, Letthe temperature dependent fluid viscosity $\eta$ is given by Reynolds model (Farooq, *et al.,* 2013). The dimensionless form of this model is

$$\eta = e(-L\theta). \quad (33)$$

Let $L = \xi m$, where $\xi$ a small perturbation parameter (Siddiqui, *et al.*, 2013, 2006). Expand equation (33) by using the Taylor series expansion up to first order and then substitute into the equation (31), we get

$$(1 - \xi m\theta)\frac{dv}{dx} = -S_t(1-x) \quad (34)$$

Keeping in mind the end goal to explain these kind of ordinary differential equations with related boundary conditions (11)-(12), here we utilize the perturbation method by fascinating the velocity and temperature distribution approximately as

$$v(x,\xi) = \sum_{i=0}^{\infty}\xi^i v_i \text{ and } \theta(x,\xi) = \sum_{i=0}^{\infty}\xi^i\theta_i \quad (35)$$

Insert equation (35) into equation (32), (34), (11) and (12) and then solve separately at each order of approximation. By using corresponding boundary conditions, the systems of equations obtained are as following.

Zeroth order problem

$$\frac{dv_0}{dx} = -S_t(1-x) \quad (36)$$

$$\frac{d^2\theta_0}{dx^2} = S_t Br(1-x)\frac{dv_0}{dx} \quad (37)$$

$$v_0 = 1, \quad \theta_0 = 1 \quad at \quad x = 0, \quad (38)$$

$$\frac{d\theta_0}{dx} = 0 \quad at\ x = 1 \quad (39)$$

First order problem

$$\frac{dv_1}{dx} - m\theta_0\frac{dv_0}{dx} = 0 \quad (40)$$

$$\frac{d^2\theta_1}{dx^2} = S_t Br(1-x)\frac{dv_1}{dx} \quad (41)$$

$$v_1 = 0, \quad \theta_1 = 0 \quad at \quad x = 0, \quad (42)$$

$$\frac{d\theta_1}{dx} = 0 \quad at\ x = 1 \quad (43)$$

We are not considering the second order equations because of long computations. Solving Equations. (36) and (37) with the associated boundary conditions (38) and (39), we have

$$v_0 = 1 - \frac{S_t}{2}\left\{1 - (1-x)^2\right\} \quad (44)$$

$$\theta_0 = 1 - \frac{S_t^2 B_r}{12}\left\{(1-x)^4 - 1\right\} \quad (45)$$

Substituting equations (44) and (45) into equations (40) and (41) and then solving with respect to the boundary conditions (42) and (43), we obtain

$$v_1 = \frac{mS_t}{2}\left\{(1-x)^2-1\right\}+\frac{mS_t^3B_r}{72}\left\{3(1-x)^2-(1-x)^6-2\right\} \quad (46)$$

$$\theta_1 = \frac{-mS_t^2B_r}{12}\left\{(1-x)^4-1\right\}+\frac{mS_t^4B_r^2}{2016}\begin{Bmatrix}3(1-x)^8- \\ 14(1-x)^4+11\end{Bmatrix} \quad (47)$$

Inserting Equation (46)–(47) into equation (35), the perturbation solutions up to order one are:

$$v(x)=1+\frac{S_t}{2}\left\{(x-1)^2-1\right\}+\frac{m\xi S_t}{2}\left\{(1-x)^2-1\right\}-\frac{m\xi S_t^3B_r}{72}\left\{2+(x-1)^6-3(x-1)^2\right\}. \quad (48)$$

$$\theta(x)=1-\frac{S_t^2B_r}{12}\left\{(x-1)^4-1\right\}-\frac{mS_t^2\xi B_r}{12}\left\{(x-1)^4-1\right\}+\frac{m\xi S_t^4B_r^2}{2016}\left\{3(1-x)^8-14(1-x)^4+11\right\} \quad (49)$$

**"Volume Flow Rate"(VFR) and "Average Velocity" (AV)**

The "volume flow rate *Q"and "*average velocity $\overline{V}$ " in dimenshionless form, is specified as:

$$Q=\int_0^1 v(x)\,dx=\overline{V} \quad (50)$$

By utilizing equation number (48) in (50), we acquire,

$$Q=\overline{V}=1-\frac{S_t}{3}-\frac{m\xi S_t}{3}-\frac{m\xi S_t^3B_r}{63} \quad (51)$$

Net upward flow can be obtained by taking dimensionless average velocity $\overline{V}>0$ Equation (51) provides

$$1>\frac{S_t}{3}+\frac{m\xi S_t}{3}+\frac{m\xi S_t^3B_r}{63} \quad (52)$$

### 3.2 Solutions for Linear PTT (LPTT)Model;

Using LPTT model (8), in equation (27) and simplifying with the help of shear stress (15) and equation (25), we get

$$\frac{dv}{dx}=-\frac{\rho g}{\eta}(\delta-x)-\frac{2\varepsilon\lambda^2\rho^3g^3}{\eta^3}(\delta-x)^3 \quad (53)$$

By non-dimensionalising equation (53) by using parameters (30) after dropping '*', we get

$$\frac{dv}{dx}=-\frac{S_t}{\eta}(1-x)-\frac{2\varepsilon S_t^3De^2}{\eta^3}(1-x)^3 \quad (54)$$

Using the Taylor series expansion up to first order from equation (33) into equation (54), we get

$$(1-3\xi m\theta)\frac{dv}{dx}=-S_t(1-x)(1-2\xi m\theta)-2\varepsilon S_t^3De^2(1-x)^3 \quad (55)$$

Where $De=\frac{\lambda U}{\delta}$ is Deborah number. Substituting perturbation expansion from equation (35) into the equation (55), after separating at each order of approximation, we obtain

Zeroth order problem

$$\frac{dv_0}{dx}=-S_t(1-x)-2S_t^3De^2\varepsilon(1-x)^3 \quad (56)$$

First order problem

$$\frac{dv_1}{dx}-3m\theta_0\frac{dv_0}{dx}=2m\theta_0S_t(1-x) \quad (57)$$

Here again, we are considering up to first order because ofsecond-order contain lengthy calculations. Solving system of equations. (56) and (37) by using corresponding boundary conditions (38) and (39), we have

$$v_0(x)=1+\frac{S_t}{2}\left\{(x-1)^2-1\right\}-\frac{\varepsilon S_t^3De^2}{2}\left\{1-(1-x)^4\right\} \quad (58)$$

$$\theta_0=1-\frac{S_t^2B_r}{12}\left\{(1-x)^4-1\right\}-\frac{S_t^4De^2B_r\varepsilon}{15}\left\{(1-x)^6-1\right\} \quad (59)$$

Insert equation (58) and (59) into the system of equations (57) and (41) and then solving with respect conditions from equation (42) and (43), we acquire

$$v_1=-\frac{mS_t}{2}\left\{1-(x-1)^2\right\}+\frac{mS_t^3B_r}{72}\left\{3(x-1)^2-(x-1-)^6-2\right\}+\frac{m\varepsilon S_t^5De^2B_r}{240}\left\{-17(1-x)^8+16(1-x)^2+30(1-x)^4-29\right\}+\frac{m\varepsilon^2S_t^7B_rDe^4}{50}\left\{-2(1-x)^{10}+5(1-x)^4-3\right\}+\frac{3m\varepsilon S_t^3De^2}{2}\left\{(1-x)^4-1\right\} \quad (60)$$

$$\theta_1=\frac{mS_t^4Br^2}{672}\left\{3(1-x)^8-14(1-x)^4+11\right\}+\frac{mS_t^6De^2\varepsilon Br^2}{2700}\left\{47(1-x)^{10}-45(1-x)^4-135(1-x)^6+133\right\}-\frac{mS_t^2Br(1-x)^4}{12}+\frac{m\varepsilon^2S_t^8Br^2De^4}{9900}\left\{30(1-x)^{12}-132(1-x)^6+102\right\}-\frac{m\varepsilon S_t^4BrDe^4}{5}\left\{(1-x)^6-1\right\} \quad (61)$$

Inserting Equation (58)–(60) into equation (35), the perturbation solutions up to order one are:

$$v(x)=1+\frac{S_t}{2}\left\{(x-1)^2-1\right\}-\frac{\varepsilon S_t^3De^2}{2}\left\{1-(1-x)^4\right\}+\frac{m\xi S_t}{2}\left\{(1-x)^2-1\right\}+\frac{mS_t^3\xi B_r}{72}\left\{3(1-x)^2-(x-1)^6-2\right\}+\frac{3m\varepsilon\xi S_t^3De^2}{2}\left\{(1-x)^4-1\right\}+\frac{m\varepsilon S_t^5\xi De^2B_r}{240}\left\{-17(1-x)^8+30(1-x)^4+16(1-x)^2-29\right\}+\frac{m\varepsilon^2\xi S_t^7B_rDe^4}{50}\left\{-2(1-x)^{10}+5(1-x)^4-3\right\} \quad (62)$$

$$\theta(x)=1-\frac{S_t^2B_r}{12}\left\{(x-1)^4-1\right\}-\frac{S_t^4De^2B_r\varepsilon}{15}\left\{(x-1)^6-1\right\}+\frac{m\xi S_t^4Br^2}{672}\left\{3(x-1)^8-14(1-x)^4+11\right\}+\frac{m\xi S_t^6De^2\varepsilon Br^2}{2700}\left\{47(1-x)^{10}-45(1-x)^4-135(1-x)^6+133\right\}-\frac{m\xi\varepsilon S_t^4BrDe^4}{5}\left\{(1-x)^6-1\right\}-\frac{m\xi S_t^2Br(1-x)^4}{12}+\frac{m\xi\varepsilon^2S_t^8Br^2De^4}{9900}\left\{30(1-x)^{12}-132(1-x)^6+102\right\} \quad (63)$$

Using equation (62) in equation (50), get,

$$Q=1-\frac{S_t}{6}-\frac{2\varepsilon S_t^3De^2}{5}-\frac{m\xi S_t}{3}-\frac{mS_t^3\xi Br}{63}-\frac{11m\varepsilon\xi S_t^5De^2Br}{135}-\frac{12m\varepsilon^2\xi S_t^7BrDe^4}{275}-\frac{6m\xi\varepsilon S_t^3De^2}{5}-\frac{10\delta_1\xi m\varepsilon^2S_t^3De^2}{7} \quad (64)$$

For net upward flow,equation (51) provides

$$1>\frac{S_t}{6}+\frac{2\varepsilon S_t^3De^2}{5}+\frac{m\xi S_t}{3}+\frac{mS_t^3\xi Br}{63}+\frac{11m\varepsilon\xi S_t^5De^2Br}{135}+\frac{12m\varepsilon^2\xi S_t^7BrDe^4}{275}+\frac{6m\xi\varepsilon S_t^3De^2}{5} \quad (65)$$

### 3.3 Solutions for Quadratic PTT (QPTT) Model;

By utilizing QPTT model (9) in equation (27) and make straightforward with the help of shear stress (15) and equation (25), we get

$$\frac{dv}{dx}=-\frac{\rho g}{\eta}(\delta-x)-\frac{2\varepsilon\lambda^2\rho^3g^3}{\eta^3}(\delta-x)^3-\frac{2\delta_1\rho^5g^5\varepsilon^2\lambda^4}{\eta^5}(\delta-x)^5 \quad (66)$$

Now by using dimensionless parameters into the equation (66), after dropping ‘*’, we get

$$\frac{dv}{dx}=-\frac{S_t(1-x)}{\eta}-2\frac{\varepsilon S_t^{\,3}De^2(1-x)^3}{\eta^3}-2\delta_1\varepsilon^2S_t^{\,5}De^4\frac{(1-x)^5}{\eta^5} \quad (67)$$

By Utilizing the Taylor series up to first order from equation (33) into equation (67), we acquire

$$(1-5\xi m\theta)\frac{dv}{dx}=-S_t(1-x)(1-4\xi m\theta)-2\delta_1\varepsilon^2S_t^{\,5}De^4(1-x)^5$$
$$-2\varepsilon S_t^{\,3}De^2(1-x)^3(1-2\xi m\theta) \quad (68)$$

By using perturbation series from equation (35) into the equation (68), after sorting out at each order of approximation, we obtain

Zeroth order problem

$$\frac{dv_0}{dx}=-S_t(1-x)-2\varepsilon S_t^{\,3}De^2(1-x)^3-2\delta_1\varepsilon^2S_t^{\,5}De^4(1-x)^5 \quad (69)$$

First order problem

$$\frac{dv_1}{dx}-5m\theta_0\frac{dv_0}{dx}=4m\theta_0S_t(1-x)+4m\theta_0\varepsilon S_t^{\,3}De^2(1-x)^3 \quad (70)$$

Solving system of equations (68) and (37) by using corresponding boundary conditions (38) (39), we have

$$v_0(x)=1-\frac{S_t}{2}\left\{1-(1-x)^2\right\}-\frac{\varepsilon S_t^3De^2}{2}\left\{1-(1-x)^4\right\}-$$
$$\frac{\delta_1\varepsilon^2S_t^5De^4}{3}\left\{1-(1-x)^6\right\} \quad (71)$$

$$\theta_0=1-\frac{S_t^2B_r}{12}\left\{(1-x)^4-1\right\}-\frac{S_t^4De^2B_r\varepsilon}{15}\left\{(x-1)^6-1\right\}-$$
$$\frac{\delta_1S_t^6De^4B_r\varepsilon^2}{48}\left\{(1-x)^8-1\right\} \quad (72)$$

Insert equation (70) and (71) into the of equations (69) and (41) and then solving the system of ordinary differential equation with respect conditions from equation (42) and (43), we obtain

$$v_1=\frac{mS_t}{2}\left\{(x-1)^2-1\right\}+\frac{mS_t^3B_r}{72}\left\{3(x-1)^2-(1-x)^6-2\right\}+\frac{m\varepsilon S_t^5De^2B_r}{240}$$
$$\left\{-17(1-x)^8+16(1-x)^2+30(1-x)^4-29\right\}+\frac{5m\delta_1^2\varepsilon^3S_t^9B_rDe^6}{288}\left\{-(1-x)^{12}+3(1-x)^4-2\right\}+$$
$$\frac{m\varepsilon^2S_t^7B_rDe^4}{50}\left\{-2(1-x)^{10}+5(1-x)^4-3\right\}+\frac{3m\varepsilon S_t^3De^2}{2}\left\{(1-x)^4-1\right\}+$$
$$\frac{5\delta_1m\varepsilon^2S_t^3De^2}{3}\left\{(1-x)^6-1\right\}+\frac{m\delta_1\varepsilon^2S_t^7B_rDe^4}{1440}\left\{-123(1-x)^{10}+15(1-x)^2+200(1-x)^6-92\right\}$$
$$+\frac{m\delta_1\varepsilon^3S_t^9B_rDe^6}{144}\left\{-7(1-x)^{12}+16(1-x)^6-3(1-x)^4-6\right\}-$$
$$\frac{\delta_1^2m\varepsilon^4S_t^{11}De^8B_r}{288}\left\{-3(1-x)^{14}+2(1-x)^6+1\right\} \quad (73)$$

$$\theta_1=\frac{mS_t^{\,4}Br^2}{672}\left\{3(1-x)^8-14(1-x)^4+11\right\}+\frac{mS_t^{\,6}De^2\varepsilon Br^2}{2700}$$
$$\left\{47(1-x)^{10}-45(1-x)^4-135(1-x)^6+133\right\}$$
$$+\frac{m\delta_1\varepsilon^2S_t^{\,8}De^{4\cdot}Br^2}{177408}\left\{868(1-x)^{12}-165(1-x)^8-308(1-x)^4-395\right\}$$
$$-\frac{mS_t^{\,2}Br(1-x)^4}{12}-\frac{\delta_1m\varepsilon^2S_t^{\,6}BrDe^4}{140}\left\{(1-x)^8-1\right\}$$
$$+\frac{m\varepsilon^2S_t^{\,8}Br^2De^4}{9900}\left\{30(1-x)^{12}-132(1-x)^6+102\right\}-\frac{17m\delta_1\varepsilon^3S_t^{\,10}Br^2De^6}{87360}$$
$$\left\{80(1-x)^{14}-65(1-x)^8-364(1-x)^6+349\right\}-\frac{m\varepsilon S_t^{\,4}BrDe^4}{5}\left\{(1-x)^6-1\right\}+$$
$$\frac{m\delta_1^{\,2}\varepsilon^4S_t^{\,12}Br^2De^8}{7056}\left\{28(1-x)^6-45(1-x)^8+17\right\} \quad (74)$$

Inserting Equation (70)–(73) into equation (35), the perturbation solutions up to order one are:

$$v(x)=1-\frac{S_t}{2}\left\{1-(1-x)^2\right\}-\frac{\varepsilon S_t^3De^2}{2}\left\{1-(1-x)^4\right\}-$$
$$\frac{\delta_1\varepsilon^2S_t^5De^4}{3}\left\{1-(1-x)^6\right\}+\frac{m\xi S_t}{2}\left\{(x-x)^2-1\right\}+$$
$$\frac{mS_t^3\xi B_r}{72}\left\{3(1-x)^2-(1-x)^6-2\right\}+\frac{m\varepsilon\xi S_t^5De^2B_r}{240}$$
$$\left\{-17(1-x)^8+16(1-x)^2+30(1-x)^4-29\right\}+\frac{5m\xi\delta_1^2\varepsilon^3S_t^9B_rDe^6}{288}$$
$$\left\{-(1-x)^{12}+3(1-x)^4-2\right\}+\frac{m\xi\varepsilon^2S_t^7B_rDe^4}{50}\left\{-2(1-x)^{10}+5(1-x)^4-3\right\}$$
$$+\frac{3m\xi\varepsilon S_t^3De^2}{2}\left\{(1-x)^4-1\right\}+\frac{5\delta_1\xi m\varepsilon^2S_t^3De^2}{3}\left\{(1-x)^6-1\right\}+$$
$$\frac{m\xi\delta_1\varepsilon^2S_t^7B_rDe^4}{1440}\left\{-123(1-x)^{10}+15(1-x)^2+200(1-x)^6-92\right\}$$
$$+\frac{m\xi\delta_1\varepsilon^3S_t^9B_rDe^6}{144}\left\{-7(x-1-x)^{12}+16(x-1)^6-3(1-x)^4-6\right\}$$
$$-\frac{\delta_1^2m\xi\varepsilon^4S_t^{11}De^8B_r}{288}\left\{-3(1-x)^{14}+2(1-x)^6+1\right\} \quad (75)$$

$$\theta(x)=1-\frac{S_t^2B_r}{12}\left\{(1-x)^4-1\right\}-\frac{S_t^4De^2B_r\varepsilon}{15}\left\{(1-x)^6-1\right\}-$$
$$\frac{\delta_1S_t^6De^4B_r\varepsilon^2}{48}\left\{(1-x)^8-1\right\}+\frac{m\xi S_t^{\,4}Br^2}{672}\left\{3(1-x)^8-14(1-x)^4+11\right\}$$
$$+\frac{m\xi S_t^{\,6}De^2\varepsilon Br^2}{2700}\left\{47(1-x)^{10}-45(1-x)^4-135(1-x)^6+133\right\}$$
$$+\frac{m\delta_1\xi\varepsilon^2S_t^{\,8}De^4Br^2}{177408}\left\{868(1-x)^{12}-165(1-x)^8-308(1-x)^4-395\right\}-$$
$$\frac{m\xi S_t^{\,2}Br(1-x)^4}{12}-\frac{\delta_1\xi m\varepsilon^2S_t^{\,6}BrDe^4}{140}\left\{(1-x)^8-1\right\}$$
$$+\frac{m\varepsilon^2S_t^{\,8}Br^2De^4\xi}{9900}\left\{30(1-x)^{12}-132(1-x)^6+102\right\}.$$
$$-\frac{17m\delta_1\varepsilon^3S_t^{\,10}Br^2De^6\xi}{87360}\left\{80(1-x)^{14}-65(1-x)^8-364(1-x)^6+349\right\}$$
$$-\frac{m\varepsilon S_t^{\,4}BrDe^4\xi}{5}\left\{(1-x)^6-1\right\}+\frac{m\delta_1^{\,2}\varepsilon^4S_t^{\,12}Br^2De^8\xi}{7056}$$
$$\left\{28(1-x)^6-45(1-x)^8+17\right\} \quad (76)$$

For utilizing equation (74) in (50), we obtain,

$$Q=\vec{V}=1-\frac{S_t}{6}-\frac{2\varepsilon S_t{}^3De^2}{5}-\frac{8\delta_1\varepsilon^2S_t{}^5De^4}{21}-\frac{m\xi S_t}{3}-\frac{mS_t{}^3\xi Br}{63}-\frac{11m\varepsilon\xi S_t{}^5De^2Br}{135}-\frac{m\varepsilon^3\xi\delta_1{}^2S_t{}^9BrDe^6}{39}-\frac{19\delta_1{}^2\varepsilon^4\xi mS_t{}^{11}De^8Br}{5040}-\frac{12m\varepsilon^2\xi S_t{}^7BrDe^4}{275}-\frac{6m\xi\varepsilon S_t{}^3De^2}{5}-\frac{10\delta_1\xi m\varepsilon^2S_t{}^3De^2}{7}-\frac{509m\xi\delta_1\varepsilon^2s_t{}^7BrDe^4}{110880}-\frac{46m\xi\varepsilon^3S_t{}^9\delta_1BrDe^6}{1365} \quad (77)$$

For net upward flow of the fluid will be,

$$1>\frac{S_t}{6}+\frac{2\varepsilon S_t{}^3De^2}{5}+\frac{8\delta_1\varepsilon^2S_t{}^5De^4}{21}+\frac{m\xi S_t}{3}+\frac{mS_t{}^3\xi Br}{63}+\frac{11m\varepsilon\xi S_t{}^5De^2Br}{135}+\frac{m\varepsilon^3\xi\delta_1{}^2S_t{}^9BrDe^6}{39}+\frac{19\delta_1{}^2\varepsilon^4\xi mS_t{}^{11}De^8Br}{5040}+\frac{12m\varepsilon^2\xi S_t{}^7BrDe^4}{275}+\frac{6m\xi\varepsilon S_t{}^3De^2}{5}+\frac{10\delta_1\xi m\varepsilon^2S_t{}^3De^2}{7}+\frac{509m\xi\delta_1\varepsilon^2s_t{}^7BrDe^4}{110880}+\frac{46m\xi\varepsilon^3S_t{}^9\delta_1BrDe^6}{1365} \quad (78)$$

**Table 1. Comparisonof velocity profile with respect to *x* for the special cases of PTT fluid model, when**

$m=4, \xi=0.1, S_t=1.002, \varepsilon=0.01, De=2, Br=8, \delta_1=1$

| *x* | UCM | LPTT | QPTT |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0.2 | 0.732198 | 0.700961 | 0.700161 |
| 0.4 | 0.507883 | 0.458639 | 0.457467 |
| 0.6 | 0.342679 | 0.285578 | 0.284275 |
| 0.8 | 0.242595 | 0.183038 | 0.181687 |
| 1 | 0.209177 | 0.149188 | 0.147824 |

**Table 2. Comparison of temperature distribution with respect to *x* for the special cases of PTT fluid model, when**

$m=10, \xi=0.01, S_t=0.99, \varepsilon=0.01, De=4, Br=4, \delta_1=1$

| *x* | UCM | LPTT | QPTT |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0.2 | 1.00969 | 0.743955 | 0.737746 |
| 0.4 | 1.01429 | 0.562612 | 0.554578 |
| 0.6 | 1.016 | 0.44275 | 0.434355 |
| 0.8 | 1.01639 | 0.374724 | 0.366296 |
| 1 | 1.01642 | 0.3527 | 0.344272 |

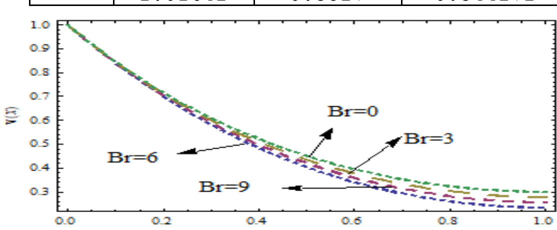


**Fig. 2: Effect of *Br*on velocity field, when**
$m=15, \xi=0.01, S_t=1.4,, \varepsilon=1, De=2$

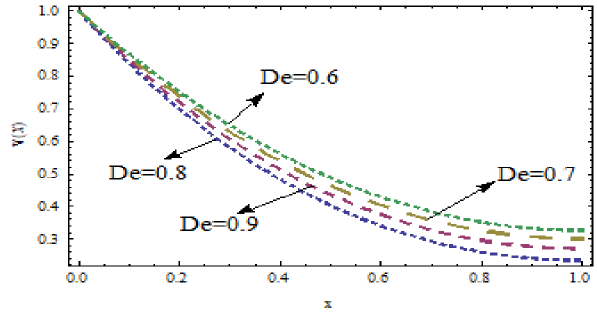


**Fig. 3: Effect of $De$ on velocity field, when**
$m=15,\ \xi=0.01,\ S_t=1.4,\ Br=10,\ \varepsilon=0.01$

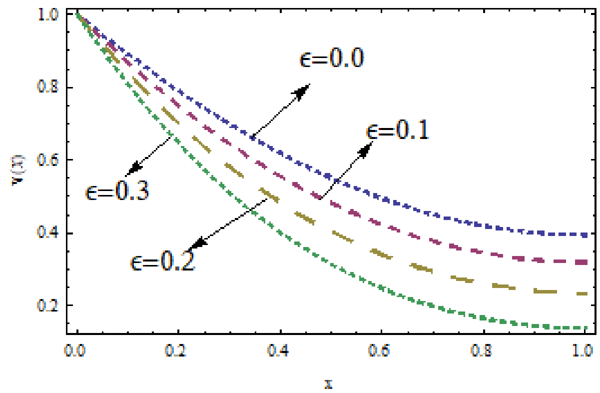


**Fig..4: Effect of $\varepsilon$ on velocity field, when**
$m=15,\ \xi=0.01,\ S_t=1.4,\ Br=10,\ De=2$

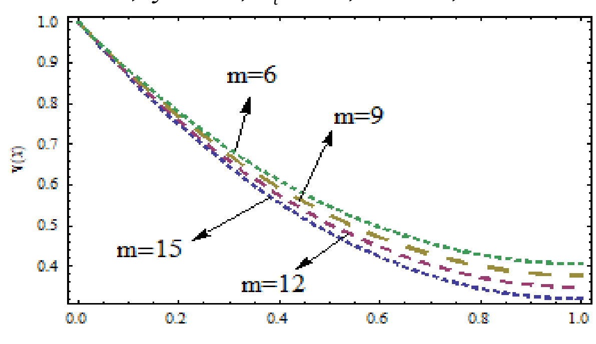


**Fig. 5: difference of *m* on velocity distribution, when**
$\varepsilon=1,\ \xi=0.01,\ S_t=1.4,\ Br=10,\ De=2$

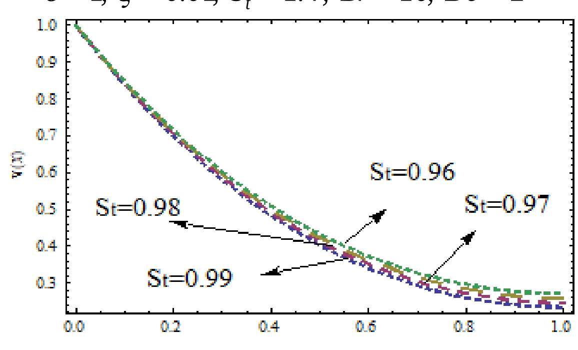


**Fig. 6: difference of $S_t$ on velocity profile, when**
$\varepsilon=1,\ \xi=0.01,\ m=15,\ Br=10,\ De=2$

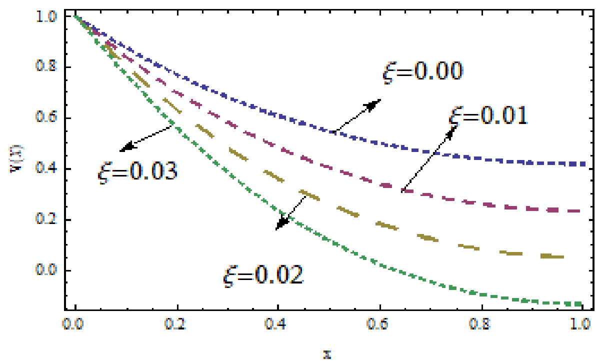


**Fig. 7: differenceof $\xi$ on velocity profile, when**
$\varepsilon=1,\ S_t=1.4,\ m=15,\ Br=10,\ De=2$

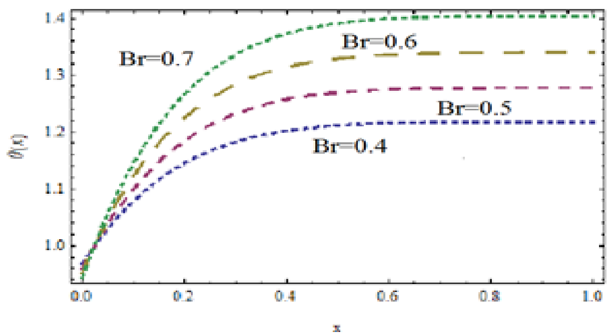


**Fig. 8: Effect of $Br$ on Temperature when**
$\varepsilon=0.01,\ S_t=0.99,\ m=10,\xi=0.1,\ De=4$

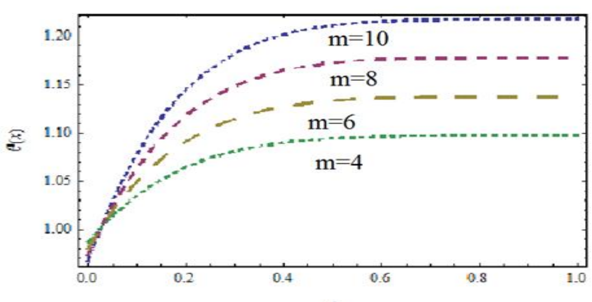


**Fig. 9: Effect of $m$ on Temperature when**
$\varepsilon=0.01,\ S_t=0.99,\ \xi=0.1,\ Br=0.4,\ De=4$

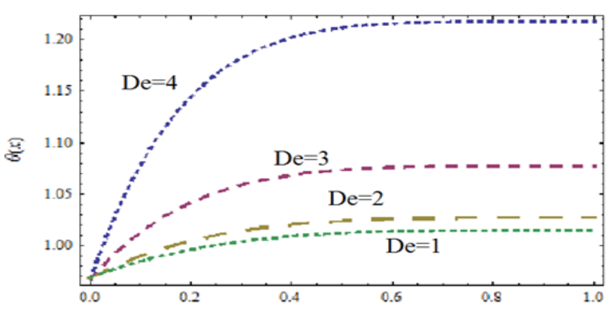


**Fig. 10: Effect of $De$ on Temperature when**
$m=10,\varepsilon=0.01,\ S_t=0.99,\ \xi=0.1,\ Br=0.4$

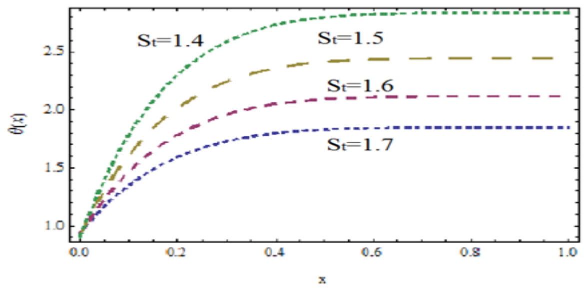


**Fig. 11: difference of $S_t$ on Temperature when**
$m=10,\varepsilon=0.01,\ De=4,\ \xi=0.1,\ Br=0.4$

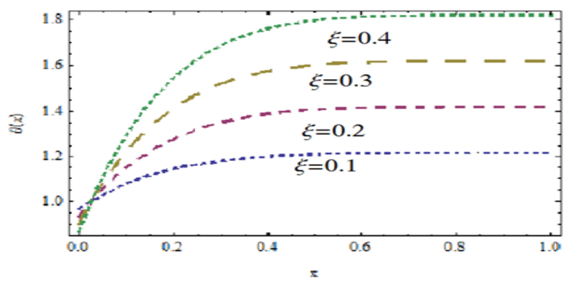


**Fig. 12: Effect of $\xi$ on Temperature when**
$m=10,\varepsilon=0.01,\ De=4,\ S_t=0.99,\ Br=0.4$

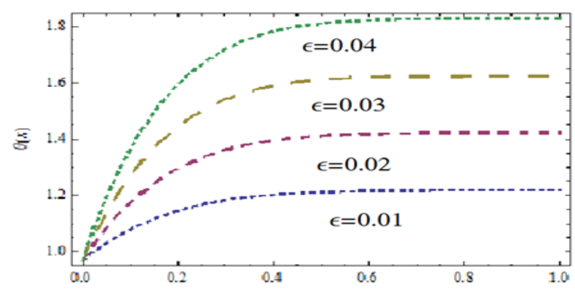


**Fig. 13: Effect of $\varepsilon$ on Temperature when**
$m=10,\xi=0.1,\ De=4,\ S_t=0.99,\ Br=0.4$ **4** **4**

## RESULTS AND DISCUSSION

In the overhead sections 1-3, we have investigatedtheoritically the thin layer flow utiling steady and incompressible Phan Thein Tanner fluid for uniform thickness with variable temperature dependent viscosity. The approximate analytic solutions have been acquired concerning velocity field and for temperature by utilizing perturbation method. The impact of distinictparameters on velocity field and for temperature is inspected graphically. The effect of the Brinkman number $Br$, Deborah number $De$, parameter represent elongational behavior $\varepsilon$, paramter $m$,Stokes number $S_t$ and perturbation parameter $\xi$ on velocity field as well as temperature distribution are observed through **(Figs. 2-13).** In the **(Figs. 2-7),** it is detected that velocity field decrease with increase of $Br, De, \varepsilon, m, S_t\ and\ \xi$. This explain that magnitude of velocity increase with the increase of all parameters.

The effect of $Br, De, \varepsilon, m, S_t\ and\ \xi$ is shown in the **(Figs 8-13).** In these Figures, one can notice that as $De, \varepsilon, m, Br\ and\ \xi$ increase, the magnitude of temperature distribution increases and decrease for the increase of $S_t$ The comparison for velocity field and temperature distribution for the special cases of PTT fluid is also given in **(Table1-2)** with specified parameters and that are declared in caption of the each table above. Results obtained in **(Table 1-2)** are calculated mathematically. Tabulated data presenting that lifting velocity field and temperature distribution of UCM fluid model is higher than LPTT fluid model and lifting velocity as well as temperature of LPTT fluid model is greater as compare to QPTT model. This indicate that lifting velocity and temperature distribution of QPTT fluid flow is slower than UCM fluid.

## 5 CONCLUDING REMARKS

Considering equation for steady, incompressible and non-isothermal thermal Phan Thein Tanner fluid for uniform thickness with variable temperature dependent viscosity on a upright belt concerning lift problem by the use of Reynold's viscosity model. By perturbation method, the resultant non-linear differential equation has been solvedfor fluid corresponding boundary conditions

The resultant non-linear differential equation (DE) has beenexplained by Perturbation method for fluid corresponding boundary conditions and that is affecting and consistent method concerning the projected problem. The velocity field, temperature distribution, average velocity and flow rate have been solved by perturbation technique. Here we have noted that UCMfluid model will uplift easily as compare to LPTT and QPTT fluid model.

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