ON REGULAR DISTRIBUTIVELY GENERATED NEAR-RINGS

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Abstract

A non-zero regular d.g. near-ring R is isomorphic to a subdirect sum of division d.g. near-rings if and only if every distributive idempotent of R is multiplicatively central.

Introduction

The theory of subdirect sum representation of rings carries over almost word for word to near-rings[3]. The result for near-rings which is analogous to Birkeff's[1] fundamental result for rings can be stated as follows: 'Every near-rings is isomorphic to a subdirect sum of subdirectly irreducible near-rings[2]. By using the technique of subdirect sum representation, it was shown[4] that every d.g. boolean near-ring is a boolean ring, and hence, is isomorphic to a subdirect sum of fields I/(2). Ligh[5] has shown that a regular near-ring $R \neq O$ is isomorphic to a subdirect sum of near-fields if and only if every idempotent of R is central. In this note we shall extend this result to a regular d.g. near-ring and we shall show that a non-zero regular d.g. near-ring R is isomorphic to a subdirect sum of division d.g. near-rings if and only if every distributive idempotent of R is multiplicatively central. We shall use some standard concepts without explicit definitions.

Preliminaries

It is assumed that reader is familiar with the basic definitions and properties of regular near-rings.[5]

Definition 1

Frehlich[2] defined an abstract distributively generated (d.g.) near-rings R to be an abstract near-ring which has a subset S of distributive elements generating (R, +).

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Definition 2

A topological near-ring is an abstract near-ring which is also a Hauslerff topological space such that the near-ring operations are continuous.

Definition 3

A topological near-ring is said to be d.g. if it has a subset S of distributive elements which generated $(R, +)$ topologically.

Throughout this note we shall assume that the term near-ring without further qualifications refers to a right near-ring with identity.

Definition 4

A topological d.g. near-ring with more than one element is said to be a division d.g. near-ring if:

(i) R has no non-trivial closed right ideals,

(ii) The set T_o (R) of all non-zero distributive elements in R forms a multiplicative group.

Clearly every division ring is a division d.g. near-ring and any division d.g. near-ring which is a ring is a division ring.The near-ring generated by the inner automorphisms of a fintite non-abelian simple group is a division d.g. near-ring.

Definition 5

An element x of a near-ring R is called central if $xy = yx$ for all y of R.

Definition 6

A near-ring R is regular if for each element x of R there exists an element y of R such that $x = xyx$.

Definition 7

A topological d.g. near-ring R is said to be a regular d.g. near-ring if

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(i) every closed right ideal of R is a d.g. right ideal,
(ii) given $t \in T(R)$ which denotes the set of all distributive elements in R, there exists $s \in T(R)$ such that $t = tst$.

Now let R be a topological d.g. near-ring with identity 1, generated by the distributive semigroup S.

Proposition 1

If R is a regular d.g. near-ring whose only non-zero distributive idempotent is 1 then R is a division d.g. near-ring.

Proof

Given $t \in T_o(R)$ there exists $s \in T_o(R)$ such that $t = tst$. Let $f = ts$ and $g = st$. Then f and g are non-zero distributive idempotents and hence $f = 1$ and $g = 1$. Thus t is invertible and $T_o(R)$ forms a multiplicative group. Clearly R has no non-trivial closed right ideals.

Proposition 2

If R is regular d.g. near-ring and ϕ is a continous homomorphism of R onto the near-ring R′ then R′ is a regular d.g. near-ring.

Proof

Let ρ' be a closed right ideal in R′. Then $\phi^{-1}(\rho')$ is a closed right ideal in R and hence is a d.g. right ideal. Thus ρ' is a d.g. right ideal in R′.

Now let $t' \in T(R')$ and $t \in T(R)$ such that $\phi(t) = t'$. Since R is a regular d.g. near-ring, there exists $s \in T(R)$ such that $t = tst$. Let $s' = \phi(s)$. Then $s' \in T(R')$ and $t' = t's't'$. Thus R′ is a regular d.g. near-ring.

Proposition 3

If R is a regular d.g. near-ring whose distributive idempotents are multiplicatively central and ϕ is a continuous homomorphism of R onto the near-ring R′ then R′ is a regular d.g. near-ring whose distributive idempotents are multiplicatively central.

Proof

We have only to prove that the distributive idempotents of R′ are multiplicatively

central. Let e′ be a distributive idempotent of R′. Then there exists t ∈ T(R) such that $\phi(t) = e'$. Since R is a regular d.g. near-ring, there exists s ∈ T(R) such that t = tst. Now ts is a distributive idempotent and hence multiplicatively central and so $t = t^2s$. But $\phi(t^2) = \phi(t)^2 = e'^2 = e' = \phi(t)$. Hence $\phi(ts) = \phi(t^2)\,\phi(s) = \phi(t) = e'$. New given $x' \in R'$ there exists x ∈ R such that $\phi(x) = x'$ and so $e'x' = \phi(ts)\,\phi(x) = \phi(tsx) = \phi(xts) = \phi(x)\,\phi(ts) = x'e'$. Thus e′ is central.

Proposition 4

If R is a non-zero subdirectly irreducible regular d.g. near-ring whose distributive idempotents are central then R is a division d.g. near-ring.

Proof

Let e ≠0,1 be a distributive idempotent of R. Let $A_e = \{y \in R : ey = 0\}$. Then since e is central, A_e is closed ideal of R. Since e ≠1, $A_e \neq 0$. Let $A = \cap A_f$ as f ranges over all distributive idempo tents not equal to 0 or 1. Since R is subdirectly irreducible, we have A ≠ 0. Let t be a non-zero distributive element in A. Then et = te = 0. Since R is regular there exists s ∈ T(R) such that t = tst and ts is a distributive idempotent. If $A_{ts} = 0$ then ts = 1 and so o = os = (et)s = o. If $A_{ts} \neq 0$, then t = tst = 0. Thus in either case we get a contradiction and so the only distributive idempotents of R are 0 and 1. Thus by Proposition 1, R is a division d.g. near-ring.

Proposition 5

A non-zero regular d.g. near-ring R is isomorphic to a subdirect sum of division d.g. near-rings if and only if every distributive idempotent of R is multiplicatively central.

Proof

The necessity of the condition follows from the fact that the only distributive idempotents of a division d.g. near-ring are 0 and 1. Now suppose that R is a regular d.g. near-ring whose distributive idempotents are central. Now every d.g. near-ring is isomorphic to a subdirect sum of subdirectly irreducible d.g. near-rings and so R is isomorphic to a subdirect sum of subdirectly irreducible d.g. near-rings R_i. By Proposition 2, 3 and 4 we have that the R_i are division d.g. near-rings and the result follows.

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