THE MIDDLE CENTRAL CHAIN

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Abstract

There exists an example of finitely generated aperiodic nilpotent group which has the upper, middle and lower central series distinct from each other.

Introduction

We define the middle central chain following Bowers [1,2] and using the presentation of finitely generated nilpotent groups in [3] and construction in [4]. We construct an example to show that the three series, that is, upper, middle and lower are distinct.

Preliminaries

In this section we deal with notation and some definition.

2.1 Definition

Let $x_1, x_2, \ldots, x_n$ be arbitrary elements of a group G. Then the commutator of x_1 and x_2 is the elements $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$ of G and for $n > 2, [[x_1, x_2, \ldots, x_{n-1}], x_n] = [x_1, \ldots, x_n]$.

2.2 Definition

Let H and K be the sub-groups of G and be the commutator sub-group function, then $\gamma(H,K) = \{ [h,k] ; (h \in H, k \in K) \}$.

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2.3 Definition

A series of sub-groups, $G = G_o \geq G1 \geq ,..., G_{i-1} \geq ... G_i = 1$ .....(
is called central series of G if G_i is a normal subgroup of G for all i and $G_i/G_{i-1} \leq Z(G_i/G$
or equivalently $\gamma(G_i, G) \leq G_{i+1}$. The length of series is γ and G_i/G_{i+1} is called a facto
series (1).

2.4 Definition

The lower central series is defined by $G = \gamma_1(G)$ and for all ordinals

$$\alpha, \gamma_\alpha G = \begin{cases} \gamma\ (\gamma_{\alpha-1}(G)) \text{ ir } \alpha\text{-1 exists} \\ \bigcap_{\theta<\alpha} \gamma_\theta(G) \quad \text{otherwise} \end{cases}$$

2.5 Definition

The upper central series is defined by $1 = Z_o(G)$ for all ordinals α, $Z_{\alpha+1}(G)$ =
$\{ x \in G; [x,y] \in Z_{\alpha+1}(G)$ and $y \in G \}$ or equivalently $Z_\alpha(G)/G_{\alpha+1}(G) \subseteq Z(G/Z_{\alpha-1}(G)$
$\alpha-1$ exists and $Z_\alpha(G) = U Z_\beta(G)$ otherwise. Thus $Z_1(G) = \{ x \in G, [x,y] \in Z_o(G)$ and y
$G \}$. That is $Z_1(G)$ is center of G.

2.6 Definition

A group G is called nilpotent if and only if it has central series. A group with cent
series of smallest length k is called nilpotent group of class k.

2.7 Definition

Let G be a group and N a normal subgroup of G. We define a subgroup $\beta(N,G)$
$< x \in G$ (there exists a positive integer m such that $x^m \in \gamma(N,G) >$. That is, $\beta(N,G)$ is
subgroup generated by those elements whose powers are in $\gamma(N,G)$.

2.8 Comment

The periodic part of $G/\gamma(N,G)$ is $\beta(N,G)/\gamma(N,G)$ and hence a subgroup of $G/\gamma(N,$

2.9 Definition

Let $\beta_o(G) = G$ and for all ordinals α.

$$\beta_\alpha(G) = \begin{cases} \beta(\beta_{\alpha-1}(G),G) \text{ if } \alpha-1 \text{ exists} \\ \bigcap_{\theta<\alpha} \beta_\theta(G) \text{ otherwise} \end{cases}$$

Lemma

Let G be a group and N a normal subgroup of G then
(i) $\beta(N,G)$ is fully invariant subgroup of G and so are $\beta_\alpha(G)$.
(ii) G/N is aperiodic and central if and only if $N \geq \beta(N,G)$.

Proof

Let θ be endomorphism of G and $a \in \beta(N,G)$. Then by the definition there exists a positive integer m such that $a^m = d$ and $d \in \gamma(N,G)$. Since $\gamma(N,G)$ is a fully invarient subgroup $a^\theta \in \beta(N,G)$ and so $\beta(N,G)$ is a fully invarient subgroup of G. Let us assume inductively that the result is true for all ordinals $\theta < \alpha$, then $\beta_\theta(G)$ is fully invarient if $\alpha-1$ exists. Therefore, if $a_1 \in \beta(\beta_{\alpha-1}(G),G)$ then $a_1^m \in \gamma(\beta_{\alpha-1}(G),G)$. Let $a_1^m = d_1$ then $(a_1^\theta)^m = d_1^\theta$ and $d_1^\theta \in \gamma(\beta_{\alpha-1}(G),G)$ and because $\gamma(\beta_{\alpha-1}(G),G)$ is a fully invarient, so $\beta(\beta_{\alpha-1}(G),G)$ is a fully invarient subgroup. If α is a limit ordinal then $\cap\beta_\alpha(G)$ is fully invarient. Hence by the Principal of Transfinite Induction, $\beta_\alpha(G)$ are fully invarient.

(ii) Let $a \in \beta(N,G)$. Then by Definition 2.7, there exists a positive integer m such that $a^m \in \gamma(N,G)$. Because $\gamma(N,G) \leq N$, a^m is in N. Thus if a does not belong to N, G/N has a periodic part, so $a \in N$. Therefore $\beta(N,G) \leq N$.

Conversely, let $\beta(N,G) \leq N$ and so G/N is central. By the First Isomorphism Theorem $G/\beta(N,G) \cong G/\gamma(N,G)/\beta(N,G)/\gamma(N,G)$. By Comment 2.8, $\beta(N,G)/\gamma(N,G)$ is periodic part of $G/\gamma(N,G)$. Therefore $G/\beta(N,G)$ is aperiodic and so G/N is aperiodic.

2.11 Definition

The middle central chain of group G is the descending central chain with aperiodic subgroups, that is, $G = \beta_0(G) \geq \beta_1(G) \geq \ldots \geq \beta_\alpha(G) \geq \ldots$

2.12 Comment

It is indeed central since the subgroups are fully invarient and $\gamma(\beta_1(G),G) \leq \beta_{1-1}(G)$ by Definition 2.7 and Leema 2.10.

§3 The middle central chain

3.1 Definition

A group G has middle central series if it has a middle central chain such that $\beta_k(G) = 1$, where k is a positive integer.

3.2 Comment

We shall now consider an example of an aperiodic nilpotent group having a middle central series different from the lower central series and than an example, having all three district series.

3.3 Example

Let G be a group having the following presentation

$G = < a, b, x, d, e, c; [a,b] = x, [x,a] = d, [x, b] = e, [d,a] = [d,b] = [d,a] = [a,c] = [b,c] = 1, [x,a] = c^v (v \in P) >$.

As in the Construction [4], we calculate the upper and lower central series. Consequently we have the upper central series,

$$1 \leq < e,c > \leq < x,e,c > \leq < a,b,x,d,e,c > = G$$

and the lower central series is

$$G = < a,b,x,d,e,c > \geq < x,e,c^v > \geq < e,c^v > > 1$$

We shall now calculate the middle series. We have $\beta_o(G) = G$. By Comment 2.8 $\beta_1(G)$ is the subgroup of G corresponding to the periodic part of $G/\gamma(\beta_o(G),G)$, under the inverse of the natural hommorophism of G onto $G/\gamma(\beta_o(G),G)$. Because $\gamma(\beta_o(G),G) = \gamma_2(G)$, the periodic part of $G/\gamma_2(G) \cong < a,b,c; [a,c] = [b,c] = [a,b] = c^v = 1, (v \in P) >$ is $< c,c^v = 1 >$. Consequently $\beta_1(G) = < x,d,e,c >$.

Similarly $\beta_2(G)/\gamma(\beta_1(G),G)$ is the periodic part of $G/\gamma(\beta_1(G),G)$, that is $G/\gamma(< x,d,e,c >, G)$. Because $\gamma(< x,d,e,c >, G)$, the periodic part is $< c,c^v = 1 >$ and so we have $\beta_2(G) = < d,e,c >$. Lastly $\beta_3(G)$ is the subgroup of G corresponding to the periodic part of $G/\gamma(\beta_2(G),G)$ under the inverse of natural homomorphism of G onto

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$G/\gamma(\beta_2(G),G)$. Because $\gamma(\beta_2(G),G) = 1$ so $\beta_3(G) = 1$.

Hence G has the middle central series.

$< a,b,x,d,e,c > \geq < x,d,e,c > \geq < d,e,c > \geq 1.$

which is identical with the upper central series but distinct from the lower central series.

3.4 Example

We shall now construct a group G which has all three series distinct from each other. Let H be the generalised direct product isomorphic to G of the construction[4]. Let B be an infinite cyclic group generated by b. We form the direct product of H and B and call it G. Therefore if $H = < x_1,\ldots, x_n, y_{\sigma 2},\ldots, y_{\sigma k}, c; [x_{i1},x_{i2}] = y_{\sigma 2}\ (i_1 < i_2),\ldots, [x_{i1}, x_{i2},\ldots, x_{ik},x_{ik+1}] = 1, y_{sk} = c^v, (v \in P) >$ and $B = <b>$. Then $G = H \times B$ with $H \cap B = 1$. Because H is nilpotent of class k, so G as homomorphic image of the direct product of nilpotent groups of class k, must be nilpotent class k.

The upper central series of G is the direct product of the corresponding upper central terms of H and B. Thus again using the calculation as in the construction[4], we have

$Z_1(G) = Z_1(H) \times B = < y_{\sigma k}, c, b >$, since B is the cyclic, the upper central series is

$$1 < < y_{\sigma k}, c, b > \leq < y_{\sigma,k-1}, y_{\sigma k}, c, b > \leq \ldots \leq < H \times B > = G.$$

We have the lower central series of G,

$$G = <H \times B> \geq y_{\sigma 2},\ldots, y_{\sigma k}, c^v > \geq \ldots < y_{\sigma,k-1}, y_{\sigma k}, c^v > \geq < y_{1k},\ldots, y_{s-1,k}, c^v > \geq 1.$$

The middle central chain can be calculated as in example.

2.9 Thus we have $G = \beta_0(G) = H \times B$. $\beta_1(G) = <$ y ∈ G there exists m a positive integer such that $y^m \in \gamma(\beta_0(G),G) > = < y_{\sigma 2}, y_{\sigma 3},\ldots, y_{\sigma,k-1}, y_{1k},\ldots, y_{s-1,k}, c >$

Let us assume inductively that

$\beta_{n-1}(G) = < y_{\sigma 2},\ldots, y_{\sigma,k-1},\ldots, y_{s-1,k}, c >$.

Thus $\beta_n(G)$ can be calculated by taking the periodic part of $G/\gamma(\beta_{n-1}(G),G)$. That is, the periodic part of $G/\gamma(\beta_{n-1}(G),G)$. Because $\gamma(\beta_{n-1}(G),G) = \gamma_n(G)$, we have the periodic

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part $\cong < c,c^v = 1 >$. Consequently $\beta_n(G) = < y_{\sigma,n+1}, \ldots, y_{1k}, \ldots, y_{s-1,k}, c >$. Thus by the principle of Induction

$$\beta_n(G) = < y_{\sigma,n+1}, \ldots, y_{\sigma,k-1}, y_{1k}, \ldots, y_{s-1,k}, c >.$$

Therefore $\beta_{k-1}(G) = < y_{1k}, \ldots, y_{s-1,k}, c >$; to calculate $B_k(G)$ we will have to calculate the periodic part of $G/\gamma\ (\beta_{k-1}(G),G)$. Because $\gamma(\beta_{k-1}(G),G) = \gamma(< y_{\sigma k},c >,G) = \gamma_{k+1}(G) = 1$. And so the periodic part of $G/\gamma\ (\beta_{k-1}(G),G) = 1$. Therefore we have $\beta_k(G) = 1$. Consequently G has the middle central series, that is

$$G \geq < y_{\sigma 2}, \ldots, y_{\sigma,k-1}, y_{1k}, \ldots, y_{s-1,k}, c > \geq \ldots\ldots \geq \ldots\ldots \geq < y_{\sigma,k-1}, y_{1k}, \ldots, y_{s-1,k}, c > \geq < y_{1k}, \ldots, y_{s-1,k}, c > \geq 1.$$

Hence the central series of G are all distinct.

References

1. J.F. Bowers, "Classes of Groups which Resemble Varieties" (preprint).
2. J.F. Bowers, "Completely Infinite Groups" (preprint).
3. A.G. Kazi, "Aperiodic Nilpotent Group" Sind. Univ. Res. Jour. (Sci. Ser.) **18**(2): (1986).
4. A.G. Kazi, "On Generalised Direct Product" Sind. Univ. Res. Jour. (Sci. Ser.) **20**(1): (1988).

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