EQUATIONS OF MOTION IN MOMENTUM SPACE

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The problem of mechanical systems moving with nonholonomic constraints which are nonlinear in the velocity components is gaining considerable importance in rational mechanics. Though nonlinear nonholonomic constraints have so far been introduced in problems by a purely mathematical device, a general theory of these constraints is quite useful. In fact, a general theory of nonholonomic systems is analytically equivalent to the problem of determining with the greatest possible accuracy the motion of a holonomic system in which the forces are not completely known, though certain first integrals are known.

The customary choices of variables used to characterise a mechanical system are (*i*) the generalised coordinates and their time derivatives and (*ii*) the generalised coordinates and momenta. In the former case the system is described by a Lagrangian L [1, 3, 4] and in the latter by the Hamiltonian H [1].

In the present paper the equations of motion of a nonholonomic mechanical system are derived by taking as variables the momenta and their time derivatives.

Let us investigate a mechanical system described by the Lagrangian

$$L=L(t;\ q_1,\ldots,q_n;\ \dot{q}_1,\ldots,\dot{q}_n),$$

where the q's are the generalised coordinates and the $\dot{q}$'s are the corresponding velocities at the time t. Suppose that the system moves subject to nonholonomic constraints of the form

$$F_l(t;\ q_1,\ldots,q_n;\ \dot{q}_1,\ldots,\dot{q}_n;\ \ddot{q}_1,\ldots,\ddot{q}_n)=0 \qquad \ldots(1)$$

$$(l=1,\ 2,\ldots,\ r<n).$$

If p_s are the generalised momenta defined by the relation

$$p_s=\frac{\partial L}{\partial \dot{q}_s} \qquad (s=1,\ 2,\ldots n) \qquad \ldots(2)$$

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then we can determine $\dot{q}$'s in terms of p's and q's and consequently $\ddot{q}$'s in terms of p's, $\dot{p}$'s and q's . Thus the equations of constraints (2) take the form

$$f_l(t; q_1, \ldots, q_n; p_1, \ldots, p_n; \dot{p}_1, \ldots, \dot{p}_n) = 0 \qquad \ldots(3)$$
$$(l = 1, 2, \ldots, r < 1).$$

Since equations (3) need not be linear in the $\dot{p}$'s, the variations of p's are defined according to N.G. Cetaev [2] by the relations

$$\sum_{s=1}^{n} \frac{\partial f_l}{\partial \dot{p}_s} \delta p_s = 0 \qquad \ldots(4)$$

The general equation of dynamics or the principle of D'Alembert—Lagrange can be written in the form

$$\sum_{s=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_s} \right) - \frac{\partial L}{\partial q_s} \right] \delta q_s = 0$$

or

$$\sum_{s=1}^{n} \left[\dot{p}_s - \frac{\partial L}{\partial q_s} \right] \delta q_s = 0 \qquad \ldots(5)$$

Now, taking a variation of the functions L, we get

$$\delta L = \sum_{s=1}^{n} \left(\frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s + \frac{\partial L}{\partial q_s} \delta q_s \right)$$

In view of equation (5) the last result becomes

$$\delta L = \sum_{s=1}^{n} \left(p_s \delta \dot{q}_s + \dot{p}_s \delta q_s \right)$$

$$= \sum_{s=1}^{n} \delta \left(p_s \dot{q}_s + \dot{p}_s q_s \right) - \sum_{s=1}^{n} \dot{q}_s \delta p_s - \sum_{s=1}^{n} q_s \delta \dot{p}_s \qquad \ldots(6)$$

If we introduce a function K defined by the relation

$$K = L - \sum_{s=1}^{n} \left(p_s \dot{q}_s + \dot{p}_s q_s \right),$$

we can write (6) in the form

$$\delta K = -\sum_{s=1}^{n} \dot{q}_s \delta p_s - \sum_{s=1}^{n} q_s \delta \dot{p}_s \qquad \ldots(7)$$

Since variations in the q's and $\dot{q}$'s do not appear on the right-hand side of (7), K may be regarded as function of the p's, $\dot{p}$'s and t. Hence

$$\delta K = \sum_{s=1}^{n} \frac{\partial K}{\partial p_s} \delta p + \sum_{s=1}^{n} \frac{\partial K}{\partial \dot{p}_s} \delta \dot{p}_s \qquad \ldots(8)$$

From (7) and (8) we get

$$\sum_{s=1}^{n} \left(\frac{\partial K}{\partial p_s} + \dot{q}_s \right) \delta p_s + \sum_{s=1}^{n} \left(\frac{\partial K}{\partial \dot{p}_s} + q_s \right) \delta \dot{p}_s = 0, \qquad \ldots(9)$$

where the δp satisfy relations (4). Employing the method of the undertermined multipliers, we obtain from (4) and (9) the relation

$$\sum_{s=1}^{n} \left(\frac{\partial K}{\partial p_s} + \dot{q}_s + \sum_{l=1}^{r} \lambda_l \frac{\partial f_l}{\partial p_s} \right) \delta p_s + \sum_{s=1}^{n} \left(\frac{\partial K}{\partial \dot{p}_s} + q_s \right) \delta \dot{p}_s = 0$$

Consequently

$$\frac{\partial K}{\partial \dot{p}_s} = -q_s \qquad \ldots(10)$$

and

$$\frac{\partial K}{\partial p_s} + \dot{q}_s + \sum_{l=1}^{r} \lambda_l \frac{\partial f_l}{\partial p_s} = 0, \quad (s = 1, 2, \ldots, n) \qquad \ldots(11)$$

Eliminating q_s from (10) and (11), we obtain the equations of motion

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{p}_s} \right) - \frac{\partial K}{\partial p_s} = \sum_{l=1}^{r} \lambda_l \frac{\partial f_l}{\partial p_s} \quad (s = 1, 2, \ldots, n) \qquad \ldots(12)$$

If the equations of constraints (3) do not contain the $\dot{p}$'s, the equations (12) assume the form

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{p}_s} \right) - \frac{\partial K}{\partial p_s} = \sum_{l=1}^{r} \lambda_l \frac{\partial f_l}{\partial p_s},$$

obtained by replacing $\frac{\partial f_l}{\partial \dot{p}_s}$ by $\frac{\partial f_l}{\partial p_s}$.

For holonomic mechanical systems the equations (12) reduce to the known equations (5)

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{p}_s}\right) - \frac{\partial K}{\partial p_s} = 0$$

References

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