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Tridiagonal Iterative Method for Linear Systems

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Abstract: In this study, we propose a tridiagonal iterative method to solve linear systems based on dominant tridiagonal entries. For solving a tridiagonal system, we incorporated the proposed method with Thomas algorithm in each step of the method. Moreover, this paper presents a comprehensive theoretical analysis, wherein we choose two well-known methods for comparison i.e., the Gauss-Seidel and Jacobi. The numerical experiment shows that our proposed iterative method is a feasible and effective method than the studied methods.

Keywords: Iterative method; tridiagonal system; Thomas algorithm, Jacobi and Gauss-Seidel

I. INTRODUCTION *AND PRELIMINARIES*

Consider the linear system

$$Ax = b, \qquad (1)$$

where $A \in R^{n\times n}$ is a non-singular matrix with dominant tridiagonal parts, i.e., the entries in the tridiagonal parts are very large compared with other entries. In some applications, such as numerical solution of differential equations [4, 6], we encounter such type of the problem in linear systems. The well-known iterative method, i.e., Gauss-Seidel and Jacobi iterative methods are not very effective for such type of systems due to special structure of the nonsingular matrix. In this study, we present an updated version of the iterative method for tridiagonal linear systems. Each step of this method is required for solving a tridiagonal system by Thomas algorithm. We provide some theoretical analysis for this new iterative method. The numerical experiment shows that our proposed iterative method is a feasible and effective method. The following are some notations and preliminaries.

Definition 1.1. Let $A \in R^{n\times n}$. If $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for all $i = 1, 2, \cdots, n$, then A is a strictly diagonally dominant matrix (SDD). If there is a positive diagonal matrix D so AD is a SDD matrix, then A is a generalized strictly diagonally dominant matrix, denoted by GDDM.

Lemma 1.1. (see [5, 14]) If A is a GDDM, then A is nonsingular and $a_{ii} \neq 0$ for $i = 1, 2, \cdots, n$.

A group of numerical methods for solving linear system $Ax = b$ is the splitting methods as follows [6, 8, 14]. Let $A = M - N$, where M is a non-singular matrix, then we have the iterative form,

$$Mx_{k+1} = Nx_k + b , \ k = 0, 1, \cdots$$

or

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b , \ k = 0, 1, \cdots \qquad (2)$$

where x_0 is a given initial vector.

Different splitting of A induce different iterative methods. The classical iterative methods include:

a) Jacobi method: $M = D$, $N = D - A$, where D is the diagonal part of A.
b) Gauss-Seidel method: $M = D + L$, $N = -U$, here D is diagonal part of A, U is strictly upper part and L is strictly lower part of triangular matrix A respectively.
c) SOR method: $M = \frac{1}{\omega} D + L$, $N = \frac{1-\omega}{\omega} D - U$,

   where ω is a parameter and DLU be as above.

Other iterative methods include AOR, two-stage iterative methods, multisplitting iterative methods, HSS method, QR method, and etc. For more details we refer to [1, 7, 9, 11, 15, 17].

We have the following results for the convergence of the iterative method (2).

Lemma 1.2. (see [5, 6, 8, 14]) The iterative method (2) is converge for any initial vector x_0 if $\rho(M^{-1}N) < 1$.

Consider the tridiagonal linear system $Ax = f$, where

$$A=\begin{pmatrix} a_1 & c_1 & & \\ b_2 & a_2 & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ & & b_n & a_n \end{pmatrix}$$

and $f=\begin{pmatrix} f_1 & f_2 & \cdots & f_n\end{pmatrix}^T$. The Thomas algorithm for solving such a system is as follows [12, Chapter 3.7] . Let the LU decomposition of A be as:

$$L=\begin{pmatrix} 1 & & & \\ \beta_2 & 1 & & \\ & \ddots & \ddots & \\ & & \beta_n & 1 \end{pmatrix},\ U=\begin{pmatrix} \alpha_1 & c_1 & & \\ & \alpha_2 & \ddots & \\ & & \ddots & c_{n-1} \\ & & & \alpha_n \end{pmatrix}.$$

By using following relations, the coefficients α_i and β_i can be computed easily.

$$\alpha_1=a_1,\ \beta_i=\frac{b_i}{\alpha_{i-1}},\ \alpha_i=a_i-\beta_i c_{i-1},\ i=1,2,\cdots,n.$$

Then the given tridiagonal system $Ax=f$ can be reduced into two bi-diagonal systems $Ly=f$ and $Ux=y$. For $Ly=f$ we have

$$y_1=f_1,\ y_i=f_i-\beta_i y_{i-1},\ i=2,\cdots,n,$$

and for $Ux=y$ we have

$$x_n=\frac{y_n}{\alpha_n},\ x_i=\frac{y_i-c_i x_{i+1}}{\alpha_i},\ i=n-1,\cdots,1.$$

The algorithm involves only $8n-7$ flops: $3(n-1)$ flops for generate the LU decomposition and $5n-4$ flops for solving the two bi-diagonal systems. It is showed in [12, 13] that when A is a DDM or SPD the algorithm is very stable. We organize the rest of the paper as follows. Section 2 gives updated version of iterative method and then some convergence analysis. In section 3 we use some numerical experiments to show the efficiency of the new iterative method. The conclusion is drawn in section 4.

II. UPDATED VERSION OF ITERATIVE METHOD

For the linear system (1), we give the new iterative method as follows. Let $A=M-N$, where M is the tridiagonal part of A and $N=M-A$, then we have the new iterative method

$$Mx_{k+1}=Nx_k+b,\ k=0,1,\cdots \quad (3)$$

where x_0 is a given initial vector.

Tridiagonal iterative method (TDI):

1. Set x_0, and $k=0$.
2. If $\|b-Ax_k\|<\varepsilon$, break; else
3. Solve the linear system (3) by Thomas algorithm.
4. Set $k=k+1$. Go to Step 2.

In each iteration of the TDI method, it needs only to solve a tridiagonal system by Thomas algorithm, since coefficient matrix is fixed in the iteration, only one decomposition is needed. The operation counts in each iteration of the new method with that of Jacobi method and Gauss-Seidel method are summarized in the Table 1. We can see that the operation counts in each iteration of the three methods are nearly the same.

Table 1:. Operation counts in each iteration of the three methods

Methods	**Form** $y=Nx_k+b$	**Solve** $Mx_{k+1}=y$	**Total counts**
Jacobi	$2n^2-2n$	n	$2n^2-n$
Gauss-Seidel	$n^2-\frac{1}{2}n$	n^2	$2n^2-\frac{1}{2}n$
TDI	$2n^2-6n$	$5n-4$	$2n^2-n$

We give convergence analysis of the new iterative method as follows.

Theorem 2.1. Let $A\in R^{n\times n}$ be a GDDM. Then the new method (3) is converge for any initial vector x_0.

Proof. Suppose, on the contrary, that the new method (3) is not converge. Then by Lemma 1.2 we have $\rho(M^{-1}N)\geq 1$. Thus there is a $\lambda\in\sigma(M^{-1}N)$ such that $|\lambda|=\rho(M^{-1}N)\geq 1$. For this λ we have $\det(\lambda I-M^{-1}N)=0$ or equivalently

$$\lambda^n\det(M^{-1})\det(M-\frac{1}{\lambda}N)=0.$$

Since $A = M - N$ is a GDDM, it is easy to verify that $M - \frac{1}{\lambda}N$ is also a GDDM. Thus $\det(M - \frac{1}{\lambda}N) \neq 0$. This contradicts with Lemma 1.1. Hence $\rho(M^{-1}N) < 1$.

By Lemma 1.2, the new method (3) is converge for any initial vector x_0.

For other class of matrices, the TDI method (3) may not converge. For example, when A is symmetric positive definite, as the following matrices,

$$\begin{pmatrix} 10 & 8 & 7 \\ 8 & 10 & 9 \\ 7 & 9 & 10 \end{pmatrix}, \begin{pmatrix} 16 & 1 & 1 \\ 1 & 1/8 & 1 \\ 1 & 1 & 1/16 \end{pmatrix},$$

it is easy to verify that $\rho(M^{-1}N) \geq 1$, thus the new method does not converge. The tridiagonal part of the second example is even singular.

Compared with Jacobi and Gauss-Seidel methods, there are situations where Gauss-Seidel and Jacobi methods converge but the updated version of iterative method(TDI) does not, and on the contrary the updated version of iterative method converge while Gauss-Seidel and Jacobi methods do not. The following examples and results summarized in Table 2 show this:

$$P = \begin{pmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{pmatrix},$$

Table 2: Convergence of the three methods

Matrix	Jacobi	Gauss-Seidel	TDI
P	diverge	diverge	converge
Q	converge	converge	diverge

III. NUMERICAL EXPERIMENTS

In this section, we test several experiments to show the effectiveness of the TDI method. We present computational results in terms of the numbers of iterations (IT) and CPU time in seconds of the updated version of iterative method with Gauss-Seidel and Jacobi methods. The iteration is stopped when the current iterate satisfies $\|b - Ax_k\|_2 < 10^{-6}$.

Experiment 1. Consider the linear system (1) with

$$A = \begin{pmatrix} 3 & -1 & & & & 1/2 \\ -1 & \ddots & & & \iddots & \\ & & 3 & -1 & & \\ & & -1 & 3 & \ddots & \\ & \iddots & & \ddots & \ddots & -1 \\ 1/2 & & & & -1 & 3 \end{pmatrix},$$

and $b = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$. This linear system is from [13]. For n = 256, we have the following results summarized in Table 3.

Table 3: Computational results of Experiment 1

Methods	IT	CPU
Jacobi	61	0.018209
Gauss-Seidel	43	0.009300
TDI	25	0.005761

From the computational results we can see that the TDI method is better than the Jacobi and Gauss-Seidel method.

Experiment 2. Consider the linear system (1) with

$$A = \begin{pmatrix} D & -I & & \\ -I & D & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & D \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix},$$

where A is of $p \times p$ blocks, D is $q \times q$ and I is the identity matrix. We take $b = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$. This linear system is from discretization of the Poisson equation by finite difference method with five-point method [6]. For different p, q, we have the following results summarized in Table 4.

Table 4: Computational results of Experiment 2

p	q	Methods	IT	CPU
16	16	Jacobi	957	0.129756
		Gauss-Seidel	480	0.062407
		TDI	483	0.050313
16	32	Jacobi	1548	1.607307
		Gauss-Seidel	775	0.917683
		TDI	773	0.380566
16	64	Jacobi	1872	4.587778
		Gauss-Seidel	938	2.871152
		TDI	933	1.508252
16	128	Jacobi	2006	15.600536
		Gauss-Seidel	1006	8.334551
		TDI	999	4.128771

From the computational results we can see that the TDI method is better than the Jacobi and Gauss-Seidel methods.
Experiment 3. Consider the linear system (1) with

$$A=\begin{pmatrix} 3 & 1 & & \\ 1 & 3 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 3 \end{pmatrix}+\frac{1}{n}\mathrm{rand}(n),$$

where rand(n) is a random matrix of size n and

$b=\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}^T$. For n = 256, we have the following

results summarized in Table 5.

Table 5: Computational results of Experiment 3

Methods	IT	CPU
Jacobi	90	0.985928
Gauss-Seidel	16	0.025497
TDI	8	0.014457

From the computational results we can see that the TDI method is better than the Jacobi and Gauss-Seidel method.

IV. CONCLUDING REMARKS

We have proposed a tridiagonal iterative method for linear systems. This method is more efficient when the system has a dominant tridiagonal part. We give some theoretical analysis for this updated version (TDI) method. Numerical experiments also show that this method is feasible and effective in some cases.

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REFERENCES

[1] Z.-Z. Bai, G.H. Golub and M.K. Ng. Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. SIAM Journal on Matrix Analysis and Applications, 24(2003), 603-626.

[2] Z.-Z. Bai, X. Yang. Continuous-time accelerated block successive overrelaxation methods for time-dependent Stokes equations. Journal of Computational and Applied Mathematics, 236(2012), 3265-3285.

[3] Z.-Z. Bai, Y.-H. Ran and L. Yuan. On approximated ILU and UGS preconditioning methods
for linearized discretized steady incompressible Navier-Stokes equations. Numerical Algorithms, 65(2014), 43-68.

[4] Z.-Z. Bai, R.H. Chan and Z.-R. Ren. On order-reducible sinc discretizations and block-diagonal preconditioning methods for linear third-order ordinary di_erential equations. Numerical Linear Algebra with Applications, 21(2014), 108-135.

[5] A. Berman, R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. Academic Press, New York, 1994.

[6] J.W. Demmel. Applied Numerical Linear Algebra. SIAM, Philadelphia, 1997.

[7] A. Frommer, D.B. Szyld. H-splitting and two-stage iterative methods. Numerische Mathematik, 63(1992), 345-356.
[8] G.H. Golub, C.F. Van Loan. Matrix Computations. Third Edition, The Johns Hopkins University Press, Baltimore and London, 1996.

[9] A. Hadjidimos. Accelerated overrelaxation method. Mathematics of Computation, 32(1978), 149-157.

[10] Jia-gan Hu. Iterative Methods for Linear Algebraic Systems (in Chinese). Series in Computational Methods, Beijing, 1991.

[11] D.P. O'Leary, R.E. White. Multi-splitting of matrices and parallel solution of linear systems. SIAM J. Alg. Disc. Meth., 6(1985), 630-640.

[12] A. Quarteroni, R. Sacco, F. Saleri. Numerical Mathematics. Springer-Verlag, Berlin, 2000.

[13] T. Sauer. Numerical Analysis. Pearson Education, Addison Wesley, 2006.

[14] R.S. Varga. Matrix Iterative Analysis. Springer-Verlag, Berlin, 2000.

[15] C.-L. Wang, R.-P. Wen and Y.-H. Bai. A new splitting and preconditioner for iteratively solving non-Hermitian positive de_nite systems. Computers and Mathematics with Applications, 65(2013), 1047-1058.

[16] R.-P. Wen, G.-Y. Meng and C.-L. Wang. Quasi-Chebyshev accelerated iteration methods based on optimization for linear systems. Computers and Mathematics with Applications 66(2013)934-942.

[17] R.-P. Wen, X.-H. Yan and C.-L. Wang. On the convergence of a new splitting iterative method for non-Hermitian positive de_nite linear systems. Applied Mathematics and Computation, 248(2014), 118-130.